Lecture 16

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Last time:

1) $\operatorname{Irr}(\mathcal{O}_\nu(\mathcal{A}_\lambda)) \leftrightarrow \operatorname{Irr}(C_\nu(\mathcal{A}_\lambda))$

2) $C_\nu(\mathcal{A}_\lambda) \simeq \underbrace{\mathbb{C} \oplus \dots \oplus \mathbb{C}}_{|Y^T|}$ ← for generic λ

$$\begin{tikzcd} C_\nu(\mathcal{A}_\lambda) \arrow[dr, "\sim"] & |Y^T| \arrow[d, phantom, "\shortparallel"] \\ & \Gamma(C_\nu(\mathcal{A}^{sh}_\lambda), Y^T) \end{tikzcd}$$

iso for generic λ

3) Conj: λ as above are complement to the (finite) union of "walls" of the form $f_i(\lambda) + s_i = 0$

will see how it helps to answer 3)

Back to 3d-mirror symmetry

alredy mentioned, see Lecture 11

Conj (Hikita)

$$\begin{tikzcd} Y \arrow[d] \arrow[r, leftrightarrow, "\text{dual}"] & Y^\vee \arrow[d] \\ X & X^\vee \end{tikzcd}$$

$$H^*(Y^\vee, \mathbb{C}) \simeq \mathbb{C}[X^{\nu(\mathbb{C}^\times)}] \qquad \nu : \mathbb{C}^\times \to T_Y$$

← generic

$$\mathbb{C}[X^{\nu(\mathbb{C}^\times)}] = C_\nu(\mathbb{C}[X])$$

iso of graded algebras

LHS grading by H^*, RHS by $\mathbb{C}^\times_\hbar$

Conj. (Hikita - Nakajima)

$$H^*_{T_{X^\vee}}(Y^\vee, \mathbb{C}) \simeq \mathbb{C}[X_{\mathfrak{h}_X}^{\nu(\mathbb{C}^\times)}]$$

$\circlearrowright$

iso of graded algebras

over $H^*_{T_{X^\vee}}(pt) = \mathbb{C}[\mathfrak{t}_{X^\vee}] = \mathbb{C}[\mathfrak{h}_X]$

Conj. (quantited HN)

$$H^*_{T_{X^\vee} \times \mathbb{C}^\times}(Y^\vee, \mathbb{C}) \simeq C_\nu(\mathcal{A}_{\hbar, \mathfrak{h}_X})$$

iso. of graded $\mathbb{C}[\mathfrak{h}_X, \hbar]$ - algebras

Moreover: $C_\nu(\mathcal{A}_{\hbar, \mathfrak{h}_X}) \to \Gamma(Y_{\mathfrak{h}}^T, C_\nu(\mathcal{A}^{sh}_{\hbar, \mathfrak{h}_X}))$

Corresponds to $H^*_{T_{X^\vee} \times \mathbb{C}^\times}(Y^\vee, \mathbb{C}) \to H^*_{T_{X^\vee} \times \mathbb{C}^\times}((Y^\vee)^{T_{X^\vee} \times \mathbb{C}^\times}, \mathbb{C})$

the fact that it becomes

iso for generic (λ, t_0) is just localization thm (Atiyah-Bott)

We see that (modulo HN conj.)

$$C_0(\mathcal{A}_{1,\lambda}) \simeq \underbrace{\mathbb{C}\oplus\cdots\oplus\mathbb{C}}_{|Y^T|} \quad \text{iff}$$

is iso

$$\begin{tikzcd}
H^*_{T_{X^\vee}\times\mathbb{C}^\times}(Y^\vee,\mathbb{C})_{(\lambda,1)} \arrow[r, "\sim"] \arrow[d, equal] & H^*\left((Y^\vee)^{T_{X^\vee}\times\mathbb{C}^\times},\mathbb{C}\right) \\
H^*\left((Y^\vee)^{(\lambda,1)},\mathbb{C}\right) \arrow[ur, "\sim"] &
\end{tikzcd}$$

iso if $(Y^\vee)^{(\lambda,1)} \overset{(*)}{=} (Y^\vee)^{T_{X^\vee}\times\mathbb{C}^\times}$

Now, set $\tilde{\Delta}_{Y^\vee} := \{ (\underset{\mathfrak{t}_{X^\vee}}{\underset{\cap}{f_i}}, \underset{\mathbb{C}}{\underset{\cap}{s_i}}) \leftarrow$ weights of $T\times\mathbb{C}^\times \curvearrowright T_pY^\vee \}$ for all $p\in(Y^\vee)^T$

Note that $(*)$ holds iff $f_i(\lambda)+s_i\neq 0$

More generally HN implies:

$$\mathrm{Comp}\left(\left(Y^{\vee}\right)^{(\lambda,1)}\right) \longleftrightarrow \mathrm{Irr}\left(\mathcal{O}_{\upsilon}(\mathcal{A}_{\lambda})\right)$$

Goal: prove quantized HN for parabolic Slodowy varieties and understand their geometry and repres. theory along the way. It turns out that we will need another realization of these varieties as slices in affine Grassmannians.

Affine Grassmannian

$G \leftarrow$ simple (reductive) / $\mathbb{C}$

$D := \mathrm{Spec}\,\mathbb{C}[[z]]$, $\mathring{D} := \mathrm{Spec}\,\mathbb{C}((z))$

a "formal" disc $\qquad$ $\uparrow$ punctured formal disc

$R \leftarrow$ commutative algebra $\rightsquigarrow$ $D_R := \mathrm{Spec}\,R[[z]]$

$\mathring{D}_R := \mathrm{Spec}\,R((z))$

category of affine schemes / $\mathbb{C}$

__Def:__ $\mathrm{Gr}_G : \mathrm{Aff}_{\mathbb{C}} \to \mathrm{Sets}^{op}$

$$\mathrm{Gr}_G(R) := \left\{ (\mathcal{P}, \sigma) \;\middle|\; \begin{array}{l} \mathcal{P} \curvearrowleft G\text{-torsor on } D_R \\ \sigma : \mathcal{P}^{triv}_{\mathring{D}_R} \xrightarrow{\sim} \mathcal{P}|_{\mathring{D}_R} \end{array} \right\}$$

__Thm__ Gr_G is represented by ind-scheme:

$$\mathrm{Gr}_G = \varinjlim X_i$$

take lim under $X_i \hookrightarrow X_j$ closed embeddings

projective schemes of finite type

Let's give an alternative description of $\mathbb{C}$-points of Gr_G:

$K := \mathbb{C}((z))$, $\mathcal{O} := \mathbb{C}[[z]]$

__Claim:__ $\mathrm{Gr}_G(\mathbb{C}) = G(K)/G(\mathcal{O})$

prf: fix a trivialization $\mathcal{P} \overset{\psi}{\simeq} \mathcal{P}^{triv}_D$

use that every G-bundle on D is trivial

$$\sigma^{-1} \circ \psi|_{\mathring{D}} \in G(K)/\underbrace{G(\mathcal{O})}_{\mathrm{Aut}(\mathcal{P}^{triv}_D)}$$

Ex 1: $G = \mathbb{C}^\times$

$$\mathbb{C}^\times(K) = \mathbb{C}((z))^\times = \mathbb{C}((z)) \setminus \{0\}, \quad \mathbb{C}^\times(\mathcal{O}) = \mathbb{C}[[z]]^\times \overset{\|}{=} \{a + z\mathbb{C}[[z]] \mid a \in \mathbb{C}^\times\}$$

$$\swarrow\!\!\swarrow$$

$$\mathbb{C}^\times(K)/\mathbb{C}^\times(\mathcal{O}) = \{z^k \mid k \in \mathbb{Z}\} \simeq \mathbb{Z}$$

Ex 2: $G = T \Rightarrow$

$$\begin{tikzcd}
\mathrm{Gr}_G \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \Lambda \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\
z^\lambda \arrow[d, phantom, "\|"] & \lambda \arrow[l, mapsto] \arrow[d, phantom, "\|"] \\
(z^{k_1}, \dots, z^{k_r}) & (k_1, \dots, k_r) \arrow[l, mapsto]
\end{tikzcd}$$

$$\underset{\uparrow}{\Lambda} = \mathrm{Hom}(\mathbb{C}^\times, T)$$

cocharacter lattice

more canonically

$$\begin{tikzcd}
\lambda \arrow[d, squiggly] & \in & T(\mathbb{C}[z^{\pm 1}]) \arrow[d, phantom, "\cap"] \\
z^\lambda & \in & T(\mathbb{C}((z)))
\end{tikzcd}$$

Ex 3: $G = GL_n \curvearrowright \mathbb{C}^n = V$

$$\mathrm{Gr}_{GL_n} = \Big\{ L \subset \overset{V \otimes_{\mathbb{C}} K}{\overset{\|}{V(K)}} \;\Big|\; L \leftarrow \text{free } \mathcal{O}\text{-module of rank } n \Big\}$$

$$\overset{\text{lattice}}{\downarrow}$$

prf. $L_0 := V(\mathcal{O}) \leftarrow$ lattice

$$\begin{tikzcd}
GL_n(K)/GL_n(\mathcal{O}) \arrow[r, "\sim"] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \{L \text{ - lattice}\} \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\
{[g]} \arrow[r, mapsto] & g(L_0)
\end{tikzcd}$$

Alternative description of Gr_G:

C ⟜ smooth curve $/\mathbb{C}$, $x \in C$

Prop: $$\mathrm{Gr}_G \simeq \left\{ (\mathcal{P}, \sigma) \;\middle|\; \begin{array}{l} \mathcal{P} ⟜ G\text{-bundle on } C \\ \sigma : \mathcal{P}^{\mathrm{triv}}_{C\setminus\{x\}} \xrightarrow{\sim} \mathcal{P}|_{C\setminus\{x\}} \end{array} \right\}$$

"prf": $C = D_x \cup C\setminus\{x\}$ ⟜ covering (fully faithful morphism)

idea!

D_x: formal disc around x

$$\mathcal{P}^{\mathrm{triv}}_{D_x} \overset{\psi}{\xrightarrow{\sim}} \mathcal{P}|_{D_x}$$ ⟜ fix a trivialization

$$\Downarrow$$

$$\sigma^{-1}|_{\mathring{D}_x} \circ \psi|_{\mathring{D}_x} \in G(K)/G(\mathcal{O})$$

$$G(\mathcal{O}) = \mathrm{Aut}(\mathcal{P}^{\mathrm{triv}}_{D_x})$$

Map in opposite direction:

$$[g] \in G(K)/G(\mathcal{O}) \rightsquigarrow g \in G(K)$$

$$\mathcal{P} ⟜ \text{glue} \quad \begin{array}{c} \mathcal{P}^{\mathrm{triv}}_{D_x} \\ \downarrow \\ D_x \end{array} \quad \text{and} \quad \begin{array}{c} \mathcal{P}^{\mathrm{triv}}_{C\setminus\{x\}} \\ \downarrow \\ C\setminus\{x\} \end{array} \quad \text{via } g$$

Rmk. one can deduce from the above taking $C = \mathbb{P}^1$

$$\mathrm{Gr}_G(\mathbb{C}) = G[z^{\pm 1}]/G[z]$$

Two decompositions of Gr_G.

Prop: (a) $\mathrm{Gr}_G = \coprod_{\lambda \in \Lambda^+} G[z^{-1}] \cdot z^{\lambda}$

(b) $\mathrm{Gr}_G = \coprod_{\lambda \in \Lambda^+} G[[z]] \cdot z^{\lambda}$ ← Cartan decomposition

prf. (a) We prove for $G = GL_n$.

Take $C = \mathbb{P}^1$, $x = 0 \in \mathbb{P}^1$

$$\mathrm{Gr}_G = \left\{ (\mathcal{F}, \sigma) \;\middle|\; \begin{array}{l} \mathcal{F} \leftarrow \text{rank } n \text{ vector bundle} \\ \sigma : \mathcal{F}^{\text{triv}}_{\mathbb{P}^1 \setminus \{0\}} \xrightarrow{\sim} \mathcal{F}|_{\mathbb{P}^1 \setminus \{0\}} \end{array} \right\}$$

Claim: $(\mathcal{F}, \sigma)$, $(\mathcal{F}', \sigma')$ lie in the same $G[z^{-1}]$-orbit iff $\mathcal{F} \simeq \mathcal{F}'$ (iso of rank n bundles)

Indeed, $G[z^{-1}] = \mathrm{Aut}(\mathcal{F}^{\text{triv}}_{\mathbb{P}^1 \setminus \{0\}}) \Rightarrow$

after we mod out by it we forget the choice of the trivialization.

Now Grothendieck thm $\Rightarrow$ every rank n vector bundle on $\mathbb{P}^1$ is $\simeq \mathcal{O}(k_1) \oplus \dots \oplus \mathcal{O}(k_n)$

↗ $k_1 \geq \dots \geq k_n$

this is exactly $z^{(k_1,\dots,k_n)}$ dominance condition

(b) Again prove for $G = GL_n$:

for $\lambda = (\lambda_1, \dots, \lambda_n)$, $z^{\lambda} = \mathrm{Span}_{\mathcal{O}} \langle z^{\lambda_1} e_1, \dots, z^{\lambda_n} e_n \rangle$

Consider any $L \subset V(K)$, $L = \mathrm{Span}_{\mathcal{O}} \langle v_1, \dots, v_n \rangle$

can assume: v_1 has greatest pole at 0.

After applying permutation in $S_n \subset GL_n(\mathcal{O})$ if needed can assume that:

$$v_1 = z^k v', \quad v' = \sum a_i e_i, \quad \begin{array}{l} a_i \in \mathbb{C}[[z]] \\ a_1 \in \mathbb{C}[[z]]^{\times} \end{array}$$

Apply the automorphism that sends $\begin{array}{l} e_i \mapsto e_i, \ i \neq 1 \\ e_1 \mapsto a_1^{-1} e_1 \end{array}$

get $v_1 = z^k \left(e_1 + \sum_{i>1} a_i e_i \right)$

then apply the automorphism $\begin{array}{l} e_1 + \sum_{i>1} a_i e_i \mapsto e_1 \\ e_i \mapsto e_i \end{array}$

get $v_1 = z^k e_1$.

Now can assume $v_2, \dots, v_n \in \mathrm{Span}_K \langle e_2, \dots, e_n \rangle$ and repeat the process.

Set $\mathrm{Gr}_G^{\lambda} := G[[z]]\cdot z^{\lambda}$

$$2\rho = \sum_{\alpha\in\Delta_+} \alpha \quad \swarrow$$

Prop: (a) $\mathrm{Gr}_G^{\lambda} \leftarrow$ smooth variety of dimension $\langle 2\rho, \lambda\rangle$

(b) $$\overline{\mathrm{Gr}_G^{\lambda}} = \bigsqcup_{\substack{\mu\le\lambda \\ \Downarrow \\ \lambda-\mu\in\mathbb{Z}_{\ge 0}\Delta_+}} \mathrm{Gr}_G^{\mu} \quad \leftarrow \text{projective}$$

(c) $$\mathrm{Gr}_G = \varinjlim \overline{\mathrm{Gr}_G^{\lambda}} \quad \leftarrow \text{ind-projective scheme}$$

prf.

(a) Let's compute the dimension of Gr_G^{λ}:

$$\mathrm{Gr}_G^{\lambda} = G[[z]] \big/ \mathrm{Stab}_{G[[z]]}\, z^{\lambda}$$

$$\mathrm{Stab}_{G[[z]]}\, z^{\lambda} = G[[z]] \cap z^{\lambda} G[[z]] z^{-\lambda}$$

$$\Downarrow$$

$$T_{z^{\lambda}} \mathrm{Gr}_G^{\lambda} = \mathfrak{g}[[z]] \big/ \mathfrak{g}[[z]] \cap \mathrm{Ad}_{z^{\lambda}}(\mathfrak{g}[[z]])$$

$$\mathfrak{g}[[z]] = \bigoplus_{\substack{\alpha\in\Delta\cup\{0\} \\ k\ge 0}} z^{k}\cdot \mathfrak{g}_{\alpha}$$

Pick $z^k \cdot x \in z^k \mathfrak{g}_\alpha$, $\mathrm{Ad}_{z^\lambda}(z^k x) = \overset{\nearrow}{z}^{k + \langle \lambda, \alpha \rangle} \cdot x$

lies in the stabilizer
iff $k + \langle \lambda, \alpha \rangle \geq 0$

So, $T_{z^\lambda} \mathrm{Gr}^\lambda = \left\{ z^k x_\alpha \,\middle|\, 0 \leq k < -\langle \alpha, \lambda \rangle \right\}$

Rmk: we also see that $G[[z]] \curvearrowright \mathrm{Gr}_G^\lambda$ through some f.d. quotient $G\left(\mathbb{C}[z]/z^N\right)$.

λ - dominant $\Rightarrow -\langle \alpha, \lambda \rangle > 0 \iff \langle \alpha, \lambda \rangle < 0 \Rightarrow \alpha \in \Delta_-$

$$\dim \mathrm{Gr}^\lambda = \sum_{\alpha \in \Delta_-} \langle -\alpha, \lambda \rangle = \sum_{\alpha \in \Delta_+} \langle \alpha, \lambda \rangle = \underline{\langle 2\rho, \lambda \rangle}$$

(b) $\leftarrow$ will not explain Exercise (deduce from $G = SL_2$)

(c) $\leftarrow$ follows from (b)

Loop rotation action

$\mathbb{C}^\times \curvearrowright \mathrm{Gr}_G$, $\quad t \cdot g(z) := g(t^{-1} z)$

Claim: (a) $\mathrm{Gr}_G^{\mathbb{C}^\times} = \coprod_{\lambda\in\Lambda^+} G\cdot z^\lambda$

$$\begin{tikzcd} G\cdot z^\lambda \simeq G/P_\lambda \\ \mathfrak{p}_\lambda = \bigoplus_{\langle\alpha,\lambda\rangle\le 0} \mathfrak{g}_\alpha \arrow[u] \end{tikzcd}$$

← partial flag variety

(b)

$$\begin{tikzcd} \mathrm{Gr}^\lambda_G = \{ & x \in \mathrm{Gr}_G \mid \overset{x_0}{\overset{\shortparallel}{\lim_{t\to\infty} t\cdot x}} \in G\cdot z^\lambda\} \arrow[dl] \arrow[d] \\ G\cdot z^\lambda = G/P_\lambda & x_0 \arrow[l, phantom, "\ni"] \end{tikzcd}$$

← affine fibration

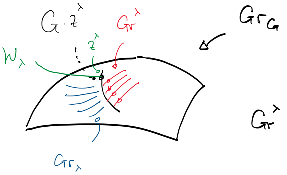
so Gr^λ is the attractor to the component

repellent to the component G/P_λ

(c)

$$\begin{tikzcd} \mathrm{Gr}_{\lambda,G} = \{x\in\mathrm{Gr}_G \mid \lim_{t\to 0} t\cdot x \in G\cdot z^\lambda\} \arrow[d] & \supset G[z^{-1}]_1\cdot z^\lambda \arrow[d] \\ G\cdot z^\lambda & \ni z^\lambda \end{tikzcd}$$

← fibers are ∞-dimensional affine spaces (schemes of ∞ type)



$$\mathrm{Gr}^\lambda \cap \mathrm{Gr}_\lambda = G\cdot z^\lambda$$

$\overline{\mathrm{Gr}}^\lambda_G$ ← singular variety.

$z^\mu \in \overline{\mathrm{Gr}}^\lambda_G$ ← pick $\mu \le \lambda$

Goal: construct a slice to $\mathrm{Gr}^\mu \subset \overline{\mathrm{Gr}^\lambda}$ at z^μ

↑

will remember singularity of $\overline{\mathrm{Gr}^\lambda}$ at z^μ

but will already be a "local" object.

Def: $\overline{\mathcal{W}}^\lambda_\mu := \overline{\mathrm{Gr}^\lambda_G} \cap \mathcal{W}_\mu$

↑

transversal slice in affine Grassmanian

Claim: $\overline{\mathcal{W}}^\lambda_\mu$ ← conical symplectic singularity

$\circlearrowleft$

$\mathbb{C}^\times$ ← loop rotation

action contracts it

to a point