Lecture 12

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Quantizations of graded Poisson algebras

$A = \bigoplus_{i\geqslant 0} A^i \leftarrow \mathbb{Z}_{\geqslant 0}$-graded Poisson algebra with

$\deg\{\ ,\ \} = 2 \quad (\{A^i, A^j\} \subset A^{i+j-2})$

Ex: $A = \mathbb{C}[X] \leftarrow$ conical symplectic singularity

Def. quantization $\mathcal{A}_\hbar$ of A is a graded associative algebra over $\mathbb{C}[\hbar]$ with $\deg \hbar = 2$ together with the identification of algebras $\underset{\mathcal{A}_0}{\overset{\shortparallel}{\mathcal{A}_\hbar/(\hbar)}} \simeq A$

s.t. for $\bar{a}, \bar{b} \in A = \mathcal{A}_0$, $a, b \in \mathcal{A}_\hbar$ lifting $\bar{a}, \bar{b}$:

$$\{\bar{a}, \bar{b}\} = \left(\overline{\frac{[a,b]}{\hbar}}\right)$$

Remarks: graded means that $[\mathcal{A}_\hbar^i, \mathcal{A}_\hbar^j] \subset \mathcal{A}_\hbar^{i+j}$

$\mathcal{A}_0$ is commutative $\Rightarrow [\mathcal{A}_\hbar^i, \mathcal{A}_\hbar^j] \subset \hbar \cdot \mathcal{A}_\hbar^{i+j-2}$

$\Downarrow$

$\frac{[a,b]}{\hbar} \in \mathcal{A}_\hbar^{i+j-2}$ is well-defined

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Ex: $A = \mathbb{C}[\overset{\overset{\mathfrak{g}}{\shortparallel}}{\mathfrak{g}^*}] \rightsquigarrow \mathcal{A}_\hbar = \underset{\shortparallel}{\mathcal{U}_\hbar(\mathfrak{g})}$

tensor algebra $\multimap$ $T^\bullet\mathfrak{g}/(x\otimes y - y\otimes x - \hbar\cdot[x,y] \mid x, y \in \mathfrak{g})$ $\leftarrow \deg \mathfrak{g} = 2$

$\underset{\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)}{\overset{}{\mathfrak{g} = \underset{\shortparallel}{\mathfrak{sl}_2}}} \Rightarrow \mathcal{U}_\hbar(\mathfrak{sl}_2) = \langle e, f, h\rangle \Big/ \left(\begin{array}{l} ef - fe = \hbar\cdot h \\ he - eh = 2\hbar e \\ hf - fh = -2\hbar f \end{array}\right)$

How to quantize

$\mathbb{C}[\mathcal{N}]$, $\mathbb{C}[T^*(G/P)]$, $\mathbb{C}[\tilde{S}(a, p)]$?

$\mathfrak{g} = \mathfrak{sl}_2$,

$$\begin{tikzcd} \mathbb{C}[\mathfrak{sl}_2] \arrow[r, squiggly] \arrow[d, two heads, "/(a^2+bc)"] & U_\hbar(\mathfrak{sl}_2) \arrow[d, two heads] \\ \mathbb{C}[\mathcal{N}] \arrow[r, squiggly] & A_\hbar(\mathcal{N}) \end{tikzcd}$$

← some quotient

Ex.

Claim: element $C := 2ef + 2fe + h^2 \in U_\hbar(\mathfrak{sl}_2)$

"Casimir element"

is central

as function on $\mathfrak{sl}_2$

For $\hbar = 0$:

$$e \rightsquigarrow \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, -\right) = b$$

$$f \rightsquigarrow \left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, -\right) = c$$

$$h \rightsquigarrow \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, -\right) = 2a$$

$$C \rightsquigarrow 2bc + 2cb + 4a^2 = 4 \cdot (a^2 + bc)$$

← exactly the equation for $\mathcal{N}$

will become clear later

$$\text{So} \quad A_{\hbar, c}(\mathcal{N}) = U_\hbar(\mathfrak{sl}_2) \big/ \left(C - \underline{c(c+2)}\,\hbar^2\right)$$

get family of quantitations of $\mathbb{C}[\mathcal{N}]$

Recall now that $\mathbb{C}[\mathcal{N}] = \mathbb{C}[T^*\mathbb{P}^1]$

How to quantize $\mathbb{C}[T^*X]$?

X ← smooth variety.

$\rightsquigarrow$

$\mathcal{D}_X$ - sheaf of differential operators on X

$\overset{..}{\|}$

$\langle \mathcal{O}_X, \mathcal{T}_X \rangle$ ← generated by functions and vector fields

Ex: $X = \mathbb{A}^1 \Rightarrow \mathcal{D}_{\mathbb{A}^1}$ ← generated by $x, \frac{\partial}{\partial x}$,

relations are $[x, \frac{\partial}{\partial x}] = 1$

Can make it homogeneous : $[x, \frac{\partial}{\partial x}] = \hbar$

In general : $[v, f] = \frac{\partial f}{\partial v} \cdot \hbar$ in $\mathcal{D}_\hbar(X)$

vector field on X

function on X

Claim: if $\pi : T^{\circ}X \to X$ projection,

then $\pi_* \mathcal{O}_{T^{\circ}X} = \mathcal{D}_\hbar(X)/(\hbar)$

$$\|$$

$$\mathbb{C}[T^{\circ}X] = \Gamma\left(\mathcal{D}_\hbar(X)/(\hbar),\, X\right)$$

$$\| \leftarrow \text{in cases that we care about}$$

global diff. operators on X $\to$ $D_\hbar(X)/(\hbar)$

So, $\mathbb{C}[T^{\circ}X] \xrightarrow{\text{quantize}} D_\hbar(X)$

Our example: $\mathbb{C}[T^{\circ}\mathbb{P}^1] \rightsquigarrow \underset{\substack{\in \\ \text{quantization of } \mathbb{C}[\mathcal{N}]}}{D_\hbar(\mathbb{P}^1)}$

How to think about $D_\hbar(\mathbb{P}^1)$:

Ham. reduction

$$T^{\circ}\mathbb{P}^1 = T^{\circ}\left((\mathbb{A}^2 \setminus \{0\})/\mathbb{C}^\times\right) = T^{\circ}(\mathbb{A}^2 \setminus \{0\}) /\!/\!/ \mathbb{C}^\times$$

$$\|$$

$$\mathbb{C}[T^{\circ}\mathbb{P}^1] = \left(\overset{x,\,y,\,\frac{\partial}{\partial x},\,\frac{\partial}{\partial y}}{\mathbb{C}[T^{\circ}\mathbb{A}^2]} \Big/ \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) \right)^{\mathbb{C}^\times}$$

vector field corresponding to the action $\mathbb{C}^\times \curvearrowright \mathbb{A}^2$, $t\cdot(x,y) = (tx,ty)$

$\deg x, y = 1$, $\deg \frac{\partial}{\partial x}, \frac{\partial}{\partial y} = -1$

$$D(\mathbb{P}^1) = D(\mathbb{A}^2) /\!/\!/ \mathbb{C}^\times = \left(\overset{\langle x,\,y,\,\frac{\partial}{\partial x},\,\frac{\partial}{\partial y}\rangle}{D(\mathbb{A}^2)} \Big/ x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - c\hbar \right)^{\mathbb{C}^\times}$$

Ex.

Claim: $D_{\hbar,c}(\overset{[x:y]}{\mathbb{P}^1}) \simeq \mathcal{A}_{\hbar,c}$

$$\begin{array}{rcl} -y\frac{\partial}{\partial x} & \leftarrow\!\shortmid & e \\ -x\frac{\partial}{\partial y} & \leftarrow\!\shortmid & f \\ y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x} & \leftarrow\!\shortmid & h \end{array}$$

$$2ef + 2fe + h^2 \mapsto 2y\frac{\partial}{\partial x}x\frac{\partial}{\partial y} + 2x\frac{\partial}{\partial y}y\frac{\partial}{\partial x} + \left(y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x}\right)^2$$

$$\|$$

$$2yx\frac{\partial}{\partial x}\frac{\partial}{\partial y} + 2\hbar y\frac{\partial}{\partial y} + 2\cancel{xy}\frac{\partial}{\partial y}\frac{\partial}{\partial x} + 2\hbar x\frac{\partial}{\partial x}$$

$$+ \left(y\frac{\partial}{\partial y}\right)^2 + \left(x\frac{\partial}{\partial x}\right)^2 - 2\cancel{yx}\frac{\partial}{\partial y}\frac{\partial}{\partial x} =$$

$$= \underbrace{\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2}_{c^2\hbar^2} + 2\hbar\cdot\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)$$

$$c^2\hbar^2 + 2\hbar\cdot c\hbar = (c^2 + 2c)\hbar^2$$

This homomorphism is induced by the map

$$\underset{\substack{\in \\ x}}{U_\hbar(\mathfrak{sl}_2)} \to \underset{\substack{\in \\ \xi_x}}{\mathcal{D}_\hbar(\mathbb{A}^2)}$$

comming from the action $\mathfrak{sl}_2 \curvearrowright \mathbb{A}^2$

$x \longmapsto \xi_x \leftarrow$ vector field by which x acts

How this generalizes to the family of

quantizations of $\mathbb{C}[T^*\mathcal{B}]$:

$$\mathbb{P}^1 = (\mathbb{A}^2 \setminus \{0\})/\mathbb{C}^\times = (SL_2/U)/\mathbb{C}^\times$$

$$\rightsquigarrow$$

$$D_{\hbar,c}(\mathbb{P}^1) = D_\hbar(SL_2/U)/\!/\!/_c\, \mathbb{C}^\times$$

$$D_{\hbar,\lambda}(\mathcal{B}) := D_\hbar(G/U)/\!/\!/_\lambda\, T \quad , \lambda \in \mathfrak{t}^* = \mathrm{Lie}\,T^*$$

$$\Big\|$$

$$\left(D_\hbar(G/U)\Big/(\xi_x - \lambda(x)\hbar^2 \mid x \in \mathfrak{t})\right)^T$$

vector field by which T acts on G/U

Can glue them all into one algebra:

$$D_{\hbar,\mathfrak{t}^*}(\mathcal{B}) = \left(D_\hbar(G/U)\right)^T$$

lives over $\mathfrak{t}^*$, specialization to λ is $D_{\hbar,\lambda}(\mathcal{B})$

Note: for $\hbar = 0$ get $D_{0,\mathfrak{t}^*} = \mathbb{C}[T^*(G/U)/T]$

$$T^*(G/U) = G \times^U \mathfrak{u}^\perp = G \times^U \mathfrak{b}$$

$$\rightsquigarrow$$

$$T^*(G/U)/T = G \times^B \mathfrak{b} = \tilde{\mathfrak{g}} = T^*_{\mathfrak{t}^*}\mathcal{B}$$

So, $D_{\hbar,\mathfrak{t}^*}(\mathcal{B})$ is the quantization of

our deformation $\mathrm{Spec}\,\mathbb{C}[T^*_{\mathfrak{t}^*}\mathcal{B}] = \mathfrak{g} \times_{\mathfrak{t}/W} \mathfrak{t}$

$$\begin{tikzcd} \mathfrak{g} \times_{\mathfrak{t}/W} \mathfrak{t} \arrow[d] \\ \mathfrak{t} = \mathfrak{t}^* \end{tikzcd}$$

How this gluing works for $\mathfrak{sl}_2$:

"$x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$"

$$\mathfrak{g} \times_{\mathfrak{t}/W} \mathfrak{t} \rightsquigarrow U_\hbar(\mathfrak{g}) \otimes_{\mathbb{C}[C,\hbar]} \mathbb{C}[t,\hbar] \simeq D_{\hbar,t}(\mathbb{P}^1)$$

Casimir

the map
$$\begin{tikzcd} \mathbb{C}[C,\hbar] \arrow[r] & \mathbb{C}[t,\hbar] \\ C \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & t(t+2\hbar) \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

If $t=\hbar c \Rightarrow C = c^2\hbar^2 + 2c\hbar^2 = (c^2+2c)\hbar^2$

and get $U_{\hbar,c}(\mathfrak{sl}_2) \simeq D_{\hbar,c}(\mathbb{P}^1)$

how to see these formulas:

have map $U_\hbar(\mathfrak{sl}_2) \to D_{\hbar,t}(\mathbb{P}^1)$

$$e \mapsto -y\frac{\partial}{\partial x}$$
$$f \mapsto -x\frac{\partial}{\partial y}$$
$$h \mapsto y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x}$$

also map
$$\begin{tikzcd} \mathbb{C}[t] \arrow[r] & D_{\hbar,t}(\mathbb{P}^1) \\ t \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

one can check that

$$C \mapsto \left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^2 + 2\hbar\cdot\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right) = t^2+2\hbar t$$

Now it's clear how to quantize $T^\circ(G/P)$, $T^\circ_{\mathfrak{z}}(G/P)$

$$\mathfrak{z} = (\mathfrak{p}/[\mathfrak{p},\mathfrak{p}])^*$$

$$T^\circ_{\mathfrak{z}}(G/P) = G \times^P [\mathfrak{p},\mathfrak{p}]^\perp \ ; \ P = M \ltimes U_P$$

The analog of $T^\circ(G/U)$ as above will

be
$$\begin{tikzcd} T^\circ(G/[P,P]) = G \times^{[P,P]} [\mathfrak{p},\mathfrak{p}]^\perp \arrow[d, "\curvearrowleft P/{[P,P]}\text{ - torsor}"] \\ G \times^P [\mathfrak{p},\mathfrak{p}]^\perp \end{tikzcd}$$

some torus

$\Downarrow$

$$D_{\hbar,\mathfrak{z}}(G/P) = \left(D_\hbar(G/[P,P])\right)^{P/[P,P]}$$

For $\lambda \in \mathfrak{z}$, fiber over λ is $D_\hbar(G/[P,P]) /\!/\!/_\lambda \, P/[P,P]$

For $\lambda = 0$ get $D_\hbar(G/P)$

Now to quantize $\tilde{S}_{\mathfrak{z}}(a,\mathfrak{p}) = T^*_{\mathfrak{z}}(G/P) /\!/\!/_\chi U$

we just do $\mathcal{A}_\hbar(\tilde{S}_{\mathfrak{h}}(a,p)) := D_{\hbar,\mathfrak{h}}(G/P)/\!\!/\!\!/_{\chi}\, U$

$\nearrow$ people call them "parabolic finite W-algebras"

$\uparrow$ again family of algebras over $\mathfrak{h}$

Category $\mathcal{O}$ for $\mathfrak{sl}_2$

$$\mathfrak{sl}_2 = \mathfrak{g} = \underset{\langle f\rangle}{\underset{\shortparallel}{\mathfrak{n}_-}} \oplus \underset{\langle h\rangle}{\underset{\shortparallel}{\mathfrak{t}}} \oplus \underset{\langle e\rangle}{\underset{\shortparallel}{\mathfrak{n}_+}}$$

Set $\hbar = 1$

Def. $\mathcal{O}(\mathcal{U}_c(\mathfrak{sl}_2)) \leftarrow$ category of $\overset{\text{finitely generated}}{\vee}$ $\mathcal{U}_c(\mathfrak{sl}_2)$-modules

M s.t.
1) e acts loc. nilpotently on M
2) h acts with finite dim. generalized eigenspaces

Example of a module in cat $\mathcal{O}$:

$$\lambda \in \mathbb{C} \rightsquigarrow \mathfrak{b} \curvearrowright \mathbb{C}_\lambda \leftarrow \begin{matrix} h\cdot 1 = \lambda h \\ e \cdot 1 = 0 \end{matrix}$$

$$\Big\downarrow$$

$$\Delta_\lambda := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda$$

$\nwarrow$ when lies in $\mathcal{O}(\mathcal{U}_c(\mathfrak{sl}_2))$?

$$C = 2ef + 2fe + h^2 \quad \text{should act by} \quad c(c+2)$$

$$\shortparallel$$

$$4fe + h^2 + 2h$$

$$C\cdot 1 = \lambda^2 + 2 \Rightarrow \lambda^2 + 2 = c^2 + 2 \Rightarrow \underline{\underline{\lambda = \pm c}}$$

Claim Δ_λ has the unique irred. quotient L_λ.

$$\mathrm{Irr}(\mathcal{O}(\mathcal{U}_c)) = \{L_\lambda \mid \lambda = \pm c\}$$

$$\Downarrow$$

$$\dim K_0(\mathcal{O}(\mathcal{U}_c)) = \#\{\lambda \mid \lambda = \pm c\} \in \{1,2\}$$

$\uparrow$ "size of our category"

See that for $c \neq 0$, $\mathcal{O}(\mathcal{U}_c)$ has $\underline{\underline{\text{two}}}$ irred so we get some "nonflat" family

for $c = 0$, $\mathcal{O}(\mathcal{U}_c)$ has one

$$\mathcal{O}(\mathcal{U}_c) \overset{\text{is}}{=} \text{Vect}$$

One can define category $\mathcal{O}(\mathcal{A}_\lambda)$ for quantization of arbitrary conical singularity.

The size of this category $(= \# \operatorname{Irr}(\mathcal{O}(\mathcal{A}_\lambda)))$ will depend on λ in a nontrivial way

It turns out that: (will discuss)

certain f.d. algebra

commutative (in all examples that we will consider)

$$\operatorname{Irr}(\mathcal{O}(\mathcal{A}_\lambda)) \xleftrightarrow{\text{natural}} \operatorname{Irr}(\underbrace{C(\mathcal{A}_\lambda)}_{\text{Cartan subquotient}})$$

$$C(\mathcal{A}_\lambda) := \mathcal{A}_{\lambda,0} \Big/ \sum \mathcal{A}_{\lambda,-i}\mathcal{A}_{\lambda,i}$$

grading is induced by generic $\mathbb{C}^\times \to T \curvearrowright \mathcal{A}_\lambda$

$\Downarrow$

$$\# \operatorname{Irr}(\mathcal{O}(\mathcal{A}_\lambda)) = \dim\left(C(\mathcal{A}_\lambda)/\text{rad}\right)$$

For $\mathfrak{sl}_2$: $\mathcal{U}_c = \mathcal{U}(\mathfrak{sl}_2) \big/ (2ef + 2fe + h^2 - c(c+2))$

grading is induced by $[h, -]$ i.e.
$\deg e = 2$
$\deg f = -2$
$\deg h = 0$

$\Downarrow$

$$C(\mathcal{U}(\mathfrak{sl}_2)) = \mathbb{C}[h], \quad 2ef + 2fe + h^2 - c(c+2) = \underset{\text{dies in } C}{2fe} + h(h+2) - c(c+2)$$

$\Downarrow$

$$C(\mathcal{U}_c) = \mathbb{C}[h] \big/ h(h+2) - c(c+2)$$

$$\simeq \mathbb{C} \oplus \mathbb{C} \quad \text{if } c \neq 0$$

$$\simeq \mathbb{C}[h] \big/ h(h+2) - 2 = \mathbb{C}[h] \big/ (h-1)^2 \quad \text{if } c = 0$$

Back to mirror symmetry

$$\begin{tikzcd} Y \arrow[d] & & Y^\vee \arrow[d, squiggly] \\ X & & X^\vee \arrow[from=1-1, to=2-3, phantom, "\multimap"] \end{tikzcd}$$

quantized Hikita-Nakajima conjecture

$$C(\mathcal{A}_{\hbar,\mathfrak{h}_X}(X)) \simeq H^*_{T_{X^\vee}\times\mathbb{C}^\times_\hbar}(Y^\vee)$$

$$\Downarrow$$

$$\begin{tikzcd} C(\mathcal{A}_{1,\lambda}(X)) \arrow[d, "/\mathrm{rad}"] \arrow[r, phantom, "\simeq"] & H^*_{T_{X^\vee}\times\mathbb{C}^\times_\hbar}(Y)_{(\lambda,1)} = H^*(Y^{(\lambda,1)}) \arrow[d, "/\mathrm{rad}"] \\ C(\mathcal{A}_{1,\lambda}(X))/\mathrm{rad} \arrow[r, phantom, "\simeq"] & H^0(Y^{(\lambda,1)}) \end{tikzcd}$$

Q

$$\Downarrow$$

$$\underset{\text{“algebra”}}{\underset{\uparrow}{\mathrm{Irr}(\mathcal{O}(\mathcal{A}_{1,\lambda}))}} \longleftrightarrow \underset{\text{“geometry”}}{\underset{\uparrow}{\mathrm{Comp}((Y^\vee)^{(\lambda,1)})}}$$