Lecture 10

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Last time formulated the following proposition and said couple words about the proof.

Prop.

$$Y = T^*(G/P) \to \operatorname{Spec} \mathbb{C}[\tilde{\mathbb{O}}] = X$$

is a conical symplectic resolution of sing.

Where the conical action comes from?

Enough to construct the action $\mathbb{C}^\times \curvearrowright \tilde{\mathbb{O}}$.

We have morphisms: $\tilde{\mathbb{O}} \to \mathbb{O} \hookrightarrow \overline{\mathbb{O}}$

conical action here sends $x \mapsto t^2 x$

Let's pick $e \in \mathbb{O} \rightsquigarrow h \in \mathfrak{g}$ s.t. $[h, e] = 2e$

Consider the corresponding 1-param subgroup $\gamma_h : \begin{array}{ccc} \mathbb{C}^\times & \to & G \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ t & \mapsto & t^h \end{array}$

Then $\operatorname{Ad}_{\gamma_h(t)}(e) = t^2 e$; $\mathbb{O} = G \cdot e$

Lemma: the $\mathbb{C}^\times$-action on $\mathbb{O}$ is given by

$$t.(ge) = g(te)$$ ← adjoint action everywhere

prf. only need to check that the RHS does not depend on the choice of g in $Z_G(e)$-orbit.

Namely: $\mathbb{O} = G/Z_G(e)$ and we want to act by right multiplication via $\gamma_h(t)$.

For that to make sense, need $\gamma_h(t)$ to normalize $Z_G(e)$: $g \in Z_G(e) \rightsquigarrow \gamma_h(t) g \gamma_h(t^{-1}) \overset{???}{\in} Z_G(e)$

$$\mathrm{Ad}_{\gamma_h(t) g \gamma_h(t^{-1})}(e) = t^2 \cdot t^{-2} \cdot e = e \checkmark$$

Now, let's extend the $\mathbb{C}^\times$-action above from $\mathbb{O}$ to $\widetilde{\mathbb{O}}$.

Fix $\tilde{e} \in \widetilde{\mathbb{O}}$ any lift of e

finite index subgroup

Then $\widetilde{\mathbb{O}} = G/H$, $H := Z_G(\tilde{e}) \subset Z_G(e)$

Claim: $\gamma_h(t)$ normalizes H $\forall t \in \mathbb{C}^\times$.

Q

prf: ① $H^\circ = Z_G(e)^\circ$ is normalized by $\gamma_h(t)$
(use $\mathbb{C}^\times$- connected)

② Pick $g \in H$, enough to check that $\gamma_h(t) g \gamma_h(t)^{-1} \in g H^\circ \subset H$. That is clear since gH° is precisely connected component of g in $Z_G(e)$ and our action preserves connected components.

Now we can define the action on $\widetilde{\mathbb{O}}$:

$$t.(g\tilde{e}) = g(t\tilde{e}).$$

This action induces the desired $\mathbb{C}^\times$- action on $\mathbb{C}[\widetilde{\mathbb{O}}] \Rightarrow$ on $X = \operatorname{Spec} \mathbb{C}[\widetilde{\mathbb{O}}]$

Why the action is conical:

recall that this is normalization

$\mathbb{O} \hookrightarrow \overline{\mathbb{O}}$

$$\begin{tikzcd} \mathbb{C}[\overline{\mathbb{O}}] \arrow[r, phantom, "\subset"] \arrow[r, bend right] & \mathbb{C}[\mathbb{O}] \arrow[r, phantom, "\subset"] \arrow[r, bend right] & \mathbb{C}[\widetilde{\mathbb{O}}] \end{tikzcd}$$

integral extensions, $\mathbb{C}^\times$-equivariant

Arready know that $\mathbb{C}^\times \curvearrowright \overline{\mathbb{O}}$ is conical =>

$$\Rightarrow \begin{cases} \mathbb{C}[\overline{\mathbb{O}}]_0 = \mathbb{C} \\ \mathbb{C}[\overline{\mathbb{O}}]_i = 0 \quad \text{for } i < 0 \end{cases}$$

Let's prove that $\mathbb{C}[\widetilde{\mathbb{O}}]_i = 0$ for $i < 0$.

Pick $f \in \mathbb{C}[\widetilde{\mathbb{O}}]_i$, we know $\exists\, a_0, \dots, a_{k-1} \in \mathbb{C}[\overline{\mathbb{O}}]$

s.t. $f^k + a_{k-1} f^{k-1} + \dots + a_0 = 0$. (*)

By passing to deg ik component can assume that (*) is homogeneous $\Rightarrow \deg a_j = (k-j)i < 0$

$\Downarrow$

$a_j = 0$

$\Leftarrow$

$f^k = 0 \Rightarrow f = 0$ ($\mathbb{C}[\widetilde{\mathbb{O}}]$ - reduced)

Ex.

Similarly, one can check that $\mathbb{C}[\widetilde{\mathbb{O}}]_0 = \mathbb{C}$

So $Y = T^*(G/P) \xrightarrow{\pi} \operatorname{Spec} \mathbb{C}[\tilde{\mathbb{O}}] = X$ ↖ conical

Recall: if $\mathfrak{g} = \mathfrak{sl}_n \Rightarrow X = \overline{\mathbb{O}}$

In general, we see covers of orbits.

It turns out that these covers dissapear (for arbitrary $\mathfrak{g}$) if we deform π.

Universal deformation of π.

some fixed Borel

$$\begin{tikzcd} Y = T^*\mathcal{B} = G \times^{B_0} \mathfrak{n}_0 = \{(x,\mathfrak{b}) \mid x \in \mathfrak{n}\} \arrow[d] \\ X = \mathcal{N} \end{tikzcd}$$

How to deform Y?

$$Y \rightsquigarrow \tilde{\mathfrak{g}} := \{(x,\mathfrak{b}) \mid x \in \mathfrak{b}\} = G \times^{B_0} \mathfrak{b}_0$$

Note:

$$\begin{tikzcd} \tilde{\mathfrak{g}} \arrow[d] & \ni & (x,\mathfrak{b}) \arrow[d, mapsto] \\ \mathfrak{t} = \mathfrak{b}/\mathfrak{n} & \ni & {[x]} \end{tikzcd}$$

↖ fiber over 0 is $T^*\mathcal{B}$

In type A: $$\begin{tikzcd}[column sep=small] \widetilde{\mathfrak{sl}}_n \arrow[d] & = & \{(F_\bullet, x) \arrow[d, mapsto] \mid x(F_i) \subset F_i\} \\ \mathfrak{t} & \ni & (\lambda_1, \dots, \lambda_n), \quad F_i/F_{i-1} \xrightarrow{x = \cdot\lambda_i} F_i/F_{i-1} \end{tikzcd}$$

How to similarly deform $\mathcal{N}$?

Guess: $\mathcal{N} \rightsquigarrow \mathfrak{g}$.

Indeed have a map $\tilde{\mathfrak{g}} \xrightarrow{\tilde{\pi}} \mathfrak{g}$.

NOT birational!

Reason: take generic semi-simple $s \in \mathfrak{t} \subset \mathfrak{g}$

Weyl group of G

$$\tilde{\pi}^{-1}(s) = \{\mathfrak{b} \mid s \in \mathfrak{b}\} = \{w \mathfrak{b}_0 w^{-1} \mid w \in W\}$$

Type A: $s = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, it preserves flag

$\{0\} \subset \{v_1\} \subset \{v_1, v_2\} \subset \dots \subset \mathbb{C}^n$ (v_i – standard basis in $\mathbb{C}^n$)

but also any flag: $\{0\} \subset \{v_{\sigma(1)}\} \subset \{v_{\sigma(1)}, v_{\sigma(2)}\} \subset \dots$

where $\sigma \in S_n$ is a permutation.

So, the morphism $\tilde{\mathfrak{g}} \to \mathfrak{g}$ is generically

$|W|:1$ covers.

Why this happens: base for the deformation $\tilde{\mathfrak{g}}$ is $\mathfrak{t}$ but the base for the deformation $\mathfrak{g}$ is $\mathfrak{t}/W$!

$$\mathfrak{g} \to \mathfrak{g}/\!\!/G = \mathfrak{t}/W$$

Ex: $\mathfrak{g} = \mathfrak{sl}_n \Rightarrow$

$$\begin{tikzcd} \mathfrak{g} \arrow[r] & \mathfrak{t}/S_n \\ x \arrow[u, phantom, "\in"] \arrow[r, mapsto] & \{\text{set of eigenvalues of } x\} \arrow[u, phantom, "\in"] \end{tikzcd}$$

they are $\underline{\underline{\underline{\text{not}}}}$ ordered!

So, have diagram:

$$\begin{tikzcd} \tilde{\mathfrak{g}} \arrow[r, "|W|:1"] \arrow[d] & \mathfrak{g} \arrow[d] \\ \mathfrak{t} \arrow[r, "|W|:1"'] & \mathfrak{t}/W \end{tikzcd}$$

generically

now it's a deformation over $\mathfrak{t}$ by the $\underline{\underline{\text{def.}}}$

To fix this: $\mathcal{N}_{\mathfrak{t}} := \mathfrak{g} \times_{\mathfrak{t}/W} \mathfrak{t}$

birational morphism deforming π

$$\begin{tikzcd} \tilde{\mathcal{N}}_{\mathfrak{t}} := \tilde{\mathfrak{g}} \arrow[rr, "\pi_{\mathfrak{t}}"] \arrow[dr, "\tilde{\varphi}"'] & & \mathfrak{g} \times_{\mathfrak{t}/W} \mathfrak{t} = \mathcal{N}_{\mathfrak{t}} \arrow[dl, "\varphi"] \\ & \mathfrak{t} & \end{tikzcd}$$

Fibers over generic points $s \in \mathfrak{t}$?

Pick $s \in \mathfrak{t}$ s.t. $\alpha(s) \neq 0 \quad \forall \alpha \in \Delta$.

$$\tilde{\varphi}^{-1}(s) = \left\{ [g,x] \in G \times \mathfrak{b} \;\middle|\; x = s + \overset{n \in \mathfrak{n}}{n} \right\} \Big/ B$$

Claim:
$$\begin{tikzcd} U \arrow[r, "\overset{a}{\sim}"] & s+\mathfrak{n} \\ u \arrow[r, mapsto] & u \cdot s \end{tikzcd}$$

i.e. every element $s+\mathfrak{n}$ as above is of the form $u \cdot s$ for unique $u \in U$

prf. fix a cocharacter $\mathbb{C}^\times \xrightarrow{\gamma} T$ s.t. the conjugation action $\mathbb{C}^\times \curvearrowright U$ contracts it to $\{1\}$.

(for example $2\rho := \sum_{\alpha \in \Delta} \alpha$ works; need $\langle \gamma, \alpha \rangle > 0 \ \forall \alpha$)

The action $\mathbb{C}^\times \curvearrowright s+\mathfrak{n}$ contracts it to $\{s\}$.

COMPARE WITH LECTURE ...

So, it remains to check that:

$$\begin{tikzcd} d_1 a : \mathfrak{n} \arrow[r] & \mathfrak{n} \\ x \arrow[r, mapsto] & {[s,x]} \end{tikzcd}$$
is iso

Recall: $\mathfrak{n} = \bigoplus_{\alpha\in\Delta_+} \mathfrak{n}_\alpha$; $\mathfrak{n}_\alpha \xrightarrow{[s,-]} \mathfrak{n}_\alpha$

multiplication
by $\alpha(s) \neq 0$!

$$\text{So, } \tilde{\varphi}^{-1}(s) = G \times^B (U.s) = G \times^T s = G/T$$

Now:
$$\begin{tikzcd} \varphi^{-1}(s) \arrow[r] \arrow[d, phantom, "\parallel"] & \varphi^{-1}(s) \arrow[d, phantom, sloped, "\in"] \\ G \times^T \{s\} \arrow[r, mapsto] & g.s \end{tikzcd}$$

$$\text{So, } \varphi^{-1}(s) = G.s = \mathbb{O}_s = G/Z_G(s) = G/T!$$

We see that $\tilde{\varphi}^{-1}(s) \simeq \varphi^{-1}(s) = \mathbb{O}_s$

smooth, affine, closed
orbit

Q note that the fact that it's closed follows from our argument ✓

and $\tilde{\pi}$ is iso over all
s as above.

TIME!

Alternative way of thinking:

$$\varphi^{-1}(s) = \{x \in \mathfrak{g} \mid f(x) = f(s)\ \forall f \in \mathbb{C}[\mathfrak{g}]^G\}$$

In type A this means that eigenvalues
of X are the same as of S.

If s has distinct eigenvalues $\Rightarrow$ $x \in G.s = \mathbb{O}_s$

Upshot: $\begin{tikzcd} \tilde{\mathcal{N}} \arrow[d] \\ \mathcal{N} \end{tikzcd}$ has a deformation over $\mathfrak{t}$ $\begin{tikzcd} \tilde{\mathcal{N}}_{\mathfrak{t}} \arrow[d] \\ \mathcal{N}_{\mathfrak{t}} \end{tikzcd}$

that deforms both $\tilde{\mathcal{N}}, \mathcal{N}$ to orbit $\mathbb{O}_s$ of semi-simple regular element.

This orbit is both smooth and affine and symplectic
Q

It turns out that the same holds for arbitrary $\begin{tikzcd} T^*(G/P) = Y \arrow[d] \\ \operatorname{Spec} \mathbb{C}[\tilde{\mathbb{O}}] = X \end{tikzcd}$ and covers "disappear" after deform.

Construction: $\mathfrak{z}_X := \mathfrak{p}/[\mathfrak{p},\mathfrak{p}]$ ← equal to $\mathfrak{t}$ for $\mathfrak{p} = \mathfrak{b}$

↑ this will be our deformation space

How to think about $\mathfrak{z}_X$:

Q

Note: $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{u}_{\mathfrak{p}} \Rightarrow [\mathfrak{p},\mathfrak{p}] = [\mathfrak{m},\mathfrak{m}] \oplus \mathfrak{u}_{\mathfrak{p}}$

$\mathfrak{m} = \mathfrak{z}(\mathfrak{m}) \oplus [\mathfrak{m},\mathfrak{m}]$

↖ center of $\mathfrak{m}$

So, $\mathfrak{p}/[\mathfrak{p},\mathfrak{p}] = \mathfrak{z}(\mathfrak{m})$

$$\begin{tikzcd}
Y = G \times^P \mathfrak{p}^\perp \arrow[r, rightsquigarrow] & Y_{\mathfrak{h}_X} := G \times^P [\mathfrak{p},\mathfrak{p}]^\perp \arrow[d] & \ni [g, x] \arrow[d, mapsto] \\
& \mathfrak{z}(\mathfrak{m}) \simeq (\mathfrak{p}/[\mathfrak{p},\mathfrak{p}])^* & \ni x|_{\mathfrak{p}}
\end{tikzcd}$$

use non-degenerate pairing

$$X_{\mathfrak{h}_X} := \operatorname{Spec} \mathbb{C}[Y_{\mathfrak{h}_X}] \to \mathfrak{h}_X$$

Ex.

For $x \in \mathfrak{h}_X$, $X_x = \operatorname{Spec} \mathbb{C}[Y_x]$

fiber over x

Get:

$$\begin{tikzcd}
Y_{\mathfrak{h}_X} \arrow[rr, "\pi_{\mathfrak{h}_X}"] \arrow[dr] & & X_{\mathfrak{h}_X} \arrow[dl] \\
& \mathfrak{h}_X &
\end{tikzcd}$$

← fiber over zero is $\pi : Y \to X$

Prop. pick $s \in \mathfrak{h}_X = \mathfrak{z}(\mathfrak{m})$ s.t. $\alpha(s) \neq 0$ for any $\alpha \in \Delta \setminus \Delta_M$

roots of $\mathfrak{u}_P$

Then $Y_s \simeq \mathbb{O}_s$ ← smooth, affine!

$\Downarrow$

$$X_s = \mathrm{Spec}\, \mathbb{C}[\mathbb{O}_s] = \mathbb{O}_s \underset{\text{closed}}{\subset} \mathfrak{g}$$

Proof is a direct generalization of the argument for $\tilde{\mathcal{N}}$ that we just discussed.

Leave it as an exercise.

Idea:

$$Y_s = G \times^P (U_P \cdot s) = G \times^M \{s\} = G/M$$

$$G/M = G/Z_G(s) = \mathbb{O}_s.$$

Upshot:

$$\begin{tikzcd} Y = T^*(G/P) \arrow[d] \arrow[r, squiggly, "\text{has a deform.}"] & Y_{\mathfrak{z}(\mathfrak{m})} \arrow[d] \\ X = \mathrm{Spec}\, \mathbb{C}[\tilde{\mathbb{O}}] & X_{\mathfrak{z}(\mathfrak{m})} \end{tikzcd}$$

that deforms both Y, X to $\underset{\substack{\text{smooth affine symplectic variety} \\ \text{orbit of } s\text{-simple element}}}{G/M} = \mathbb{O}_s$

NO covers appear after we deform

The existence of this deformation is a general phenomenon.

Prop. [Namikawa]

Let $Y \xrightarrow{\pi} X$ ← be a conical symplectic resolution.

The ∃ a deformation

$$\begin{tikzcd} Y_{\mathfrak{h}_X} \arrow[r, "\pi_{\mathfrak{h}_X}"] \arrow[dr] & X_{\mathfrak{h}_X} \arrow[d] \\ & \mathfrak{h}_X \end{tikzcd}$$

over the base $\mathfrak{h}_X = H^2(Y,\mathbb{C})$

s.t. ① $Y_{\mathfrak{h}_X} \to \mathfrak{h}_X$ is symplectic

② $X_{\mathfrak{h}_X} \to \mathfrak{h}_X$ is Poisson

③ $\pi_{\mathfrak{h}_X}$ ← resolution of singularities compatible with Poisson structures

④ Fibers of $\pi_{\mathfrak{h}_X}$ over generic points of $\mathfrak{h}_X$ are smooth, affine symplectic varieties

this deformation is universal with these properties

Rem: $H^2(T^*(G/P),\mathbb{C}) = H^2(G/P,\mathbb{C}) \simeq (\mathfrak{p}/[\mathfrak{p},\mathfrak{p}])^*$

Ex.

(Same argument as with G/B works)

prove that $\mathrm{Pic}(G/P) \simeq \mathrm{Pic}_G(G/P) = \mathrm{Hom}(P,\mathbb{C}^\times)$