Khovanov homology from KLRW algebras

prepared for a 2-hour talk in Geometry & Quantum Theory Seminar (learning seminar)

Main references: [Lauda - An Introduction to diagrammatic algebra and categorified quantum sl_2], [Webster - Knot invariants and higher representation theory]

Goal:
- What is Khovanov homology?
- What is KLRW algebra? How?

Background:

- Quantum groups $U_q(\mathfrak{g})$ $\rightsquigarrow$ ribbon category $\rightsquigarrow$ Reshetikhin-Turaev functor

  ~~Rep~~ $U_q(\mathfrak{g})$-mod

$$\begin{array}{c} \mathbb{C} \\ \uparrow \\ \vdots \\ \uparrow \\ V^*\otimes V\otimes V\otimes V^* \\ \uparrow \\ \mathbb{C} \end{array}$$

link invariants!

For $U_q(sl_2)$, get Jones polynomial

$\langle O \rangle = q + q^{-1}$

$\langle \times \rangle = \langle)(\rangle - q \langle \asymp \rangle$

$\langle D_1 \sqcup D_2 \rangle = \langle D_1 \rangle \langle D_2 \rangle$

$\longrightarrow \quad (-1)^{c_-(D)} \, q^{c_+(D) - 2c_-(D)} \langle D \rangle$

does not depend on the choice of diagram D of the oriented link L

- Khovanov's idea:

$\langle O \rangle = q + q^{-1} \rightsquigarrow$ Frobenius algebra $A := \mathbb{Z}[X]/(x^2)$

$\langle \times \rangle = \langle)(\rangle - q \langle \asymp \rangle \rightsquigarrow$ skein exact triangle (mapping cone),

$$[\![\times]\!] \cong \left(\underline{[\![ )( ]\!]} \to [\![\asymp]\!]\{1\} \right)$$

$(-1)^{c_-} q^{c_+ - 2c_-} \rightsquigarrow$ degree shift $[-c_-]\{c_+ - 2c_-\}$

e.g.

<u>Khovanov homology</u> is the bigraded homology groups $\mathcal{H}^{i,j}$, whose graded Euler characteristic $\sum_{i,j} (-1)^i q^j \dim(\mathcal{H}^{i,j} \otimes \mathbb{Q})$ is the Jones polynomial.

- Extension to tangles:

The <u>Bar-Natan 2-category</u> BN has objects

1-morphisms

(formal direct sums of) flat tangles

2-morphisms (formal linear combinations of) dotted cobordisms modulo

Same construction applied to tangles give chain complexes in the additive category $\mathrm{Hom}_{BN}(n,m)$

# of end points of the tangle

~~whose~~ which is an invariant up to homotopy.

- Higher representation theory

$$\begin{tikzcd}
\mathcal{U} \arrow[rr, "2\text{-representation}"] \arrow[d, squiggly, "K_0"'] & & \mathcal{V} \in \mathrm{Cat} \arrow[d, squiggly, "K_0"] \\
U \arrow[rr, "\text{representation}"'] \arrow[urr, "\text{categorified representation}"] & & V \in \mathrm{Vect}
\end{tikzcd}$$

KLRW

categorified quantum group

quantum group

vector space

$V_{\mu_1} \otimes \cdots \otimes V_{\mu_m}$ $U_q(\mathfrak{g})$-module $\qquad \mathcal{V}_{\underline{\mu}}$

$\uparrow f(T)$ RT invariant $\rightsquigarrow$ $\uparrow$ functor

$V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$ $U_q(\mathfrak{g})$-module $\qquad \mathcal{V}_{\underline{\lambda}}$

(should recover Khovanov homology for tangles)

- sl_2, quantum, and categorified

sl_2 : traceless 2×2

has basis $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$[E,F] = H, \quad [H,E] = 2E, \quad [H,F] = -2F$$

In $U(sl_2) \longrightarrow$

$$\begin{array}{ccc} \| & \| & \| \\ EF-FE & HE-EH & HF-FH \end{array}$$

$\Big\downarrow$ quantum deformation

- $U_q(sl_2)$: $\mathbb{Q}(q)$-algebra

generators : E, F, K, K^{-1} $\quad (K \sim q^H)$

relations : $KE = q^2 EK$,

$KF = q^{-2} FK$,

$$EF - FE = \frac{K-K^{-1}}{q-q^{-1}}$$

V a fin. dim'l rep. of $U_q(sl_2)$

$\rightsquigarrow$ weight decomposition $V = \bigoplus_{n\in\mathbb{Z}} V_n$ ← eigenspaces for K, $Kv = q^n v$

$$\begin{tikzcd} V_{-N} \arrow[r, bend left, "E"] & \cdots \arrow[l, bend left, "F"] & V_{n-2} \arrow[r, bend left, "E"] & V_n \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] & V_{n+2} \arrow[l, bend left, "F"] & \cdots \arrow[r, bend left, "E"] & V_N \arrow[l, bend left, "F"] \end{tikzcd}$$

On V_n, $(EF-FE)v = [n]v$

$$\frac{q^n - q^{-n}}{q-q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{1-n}$$

- Lusztig's integral form:

  add mutually orthogonal idempotents 1_n $(n \in \mathbb{Z})$

  so that $1_n 1_m = \delta_{n,m} 1_n$,

  $$K 1_n = q^n 1_n,$$
  $$E 1_n = 1_{n+2} E = 1_{n+2} E 1_n,$$
  $$F 1_n = 1_{n-2} F = 1_{n-2} F 1_n,$$
  $$EF 1_n - FE 1_n = [n] 1_n$$

  The integral form U is the $\mathbb{Z}[q,q^{-1}]$-subalgebra spanned by products of divided powers

  this makes the R-matrix integral

  $$E^{(a)} 1_n := \frac{E^a}{[a]!} 1_n,$$
  $$F^{(b)} 1_n := \frac{F^b}{[b]!} 1_n.$$

  We can think of U as a category with

  objects: $n \in \mathbb{Z}$

  morphisms: $\mathbb{Z}[q,q^{-1}]$-modules $1_m U 1_n$
  $n \to m$

- Categorical representation:

linear maps		functors	(graded or triangulated) additive categories
$1_n : V_n \to V_n$	K_0	$\mathbb{1}_n : \mathcal{V}_n \to \mathcal{V}_n$	
$E 1_n : V_n \to V_{n+2}$	$\leftsquigarrow$	$\mathcal{E} \mathbb{1}_n : \mathcal{V}_n \to \mathcal{V}_{n+2}$	Commuting with the grading shift
$F 1_n : V_n \to V_{n-2}$		$\mathcal{F} \mathbb{1}_n : \mathcal{V}_n \to \mathcal{V}_{n-2}$	$\{1\} : \mathcal{V}_n \to \mathcal{V}_n$ (auto equivalence)

with natural isomorphisms $\mathcal{E}\mathcal{F}\mathbb{1}_n \cong \mathcal{F}\mathcal{E}\mathbb{1}_n \oplus \mathbb{1}_n^{\oplus [n]}$ for $n \geq 0$,
$\mathcal{F}\mathcal{E}\mathbb{1}_n \cong \mathcal{E}\mathcal{F}\mathbb{1}_n \oplus \mathbb{1}_n^{\oplus [-n]}$ for $n \leq 0$

- categorified quantum group

$$U \rightsquigarrow \mathcal{U}$$

1-category additive 2-category

objects $n \in \mathbb{Z}$

$1_m U 1_n \overset{K_0}{\leftsquigarrow}$ Hom categories ${}_m\mathcal{U}_n$ $\mathbb{Z}$-graded (has shift functor $\{1\}$) additive categories,

$\mathbb{Z}[q,q^{-1}]$-module

composition ${}_{n''}\mathcal{U}_{n'} \times {}_{n'}\mathcal{U}_n \longrightarrow {}_{n''}\mathcal{U}_n$

$[x\{t\}] = q^t [x]$ preserves additive structure

$[x] = [x_1] + [x_2]$ if $x = x_1 \oplus x_2$

Taking the split Grothendieck group $K_0(\mathcal{U}) := \bigoplus_{n,m} K_0({}_m\mathcal{U}_n)$

and requiring that $[x] = [x_1][x_2]$ if $x = x_1 \circ x_2$, it recovers U.

* Lusztig's canonical basis $\mathbb{B}$ of U has the property that

$$[b_x][b_y] = \sum_z m^z_{x,y} [b_z] \quad \text{for } [b_x], [b_y], [b_z] \in \mathbb{B}$$

where $m^z_{x,y} \in \mathbb{N}[q,q^{-1}]$.

This suggests $\left(\begin{array}{c}\text{indecomposable 1-morphisms of } \mathcal{U} \\ \text{up to grading shift}\end{array}\right) \longleftrightarrow \mathbb{B}$

and $b_x \circ b_y \underset{\text{2-morphism}}{=} \bigoplus_z \left(\bigoplus_{m^z_{x,y}} b_z \right)$.

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For two 1-morphisms x, y of $\mathcal{U}$, consider the graded 2Hom

$$\mathrm{HOM}_{\mathcal{U}}(x,y) := \bigoplus_{t\in\mathbb{Z}} \mathcal{U}(x\{t\}, y)$$

degree preserving 2-morphisms

Its decategorification

$$\langle [x],[y]\rangle := \mathrm{gdim}\, \mathrm{HOM}_{\mathcal{U}}(x,y) = \sum_{t\in\mathbb{Z}} q^t \dim \mathcal{U}(x\{t\}, y)$$

must induce a semilinear pairing on U, i.e. $\mathbb{Z}[q,q^{-1}]$-antilinear on the first slot, $\mathbb{Z}[q,q^{-1}]$-linear on the second.

i.e. $\langle q^t x, y\rangle = q^{-t}\langle x,y\rangle$,
$\langle x, q^{t'} y\rangle = q^{t'}\langle x,y\rangle$.

Such a form had already been defined. It satisfies

(i) $\langle\, ,\, \rangle$ is semilinear

(ii) $\langle 1_{n_1} x 1_{n_2}, 1_{n_1'} y 1_{n_2'}\rangle = 0$ for all $x,y\in U$ unless $n_1=n_1'$ and $n_2=n_2'$

(iii) $\langle ux, y\rangle = \langle x, \tau(u) y\rangle$ for all u, x, y,

where $\tau : U \to U$ is an antilinear antiautomorphism

$$q^s 1_m E^{(a)} F^{(b)} 1_n \longmapsto q^{-s-(a-b)(a-b+n)} 1_n E^{(b)} F^{(a)} 1_m,$$
$$q^s 1_m F^{(b)} E^{(a)} 1_n \longmapsto q^{-s-(a-b)(a-b+n)} 1_n F^{(a)} E^{(b)} 1_m$$

(iv) $\langle E^{(a)} 1_n, E^{(a)} 1_n\rangle = \langle F^{(a)} 1_n, F^{(a)} 1_n\rangle = \prod_{j=1}^{a} \frac{1}{1-q^{2j}}$

e.g. $\langle E 1_n, E 1_n\rangle = \frac{1}{1-q^2} = 1 + q^2 + q^4 + \cdots$,

$\langle E^2 1_n, E^2 1_n\rangle = [2]^2 \frac{1}{1-q^2}\frac{1}{1-q^4} = (1+q^{-2})\left(\frac{1}{1-q^2}\right)^2$

String notations for $\mathcal{E}$ and $\mathcal{F}$:

$1_{\mathcal{E}1_n\{t\}}$ $\qquad$ $1_{\mathcal{F}1_n\{t\}}$ $\qquad$ (grading shifts omitted in the string diagrams)

- $\mathrm{gdim}(\mathrm{HOM}_{\mathcal{U}}(\mathcal{E}1_n, \mathcal{E}1_n)) = \langle E1_n, E1_n \rangle = \frac{1}{1-q^2} = 1+q^2+q^4+\cdots$

suggests a new 2-morphism of degree 2:

$\deg\Big($ $\Big) = 2,$

and likewise

$\deg\Big($ $\Big) = 2.$

- $\mathrm{gdim}(\mathrm{HOM}_{\mathcal{U}}(\mathcal{E}\mathcal{E}1_n, \mathcal{E}\mathcal{E}1_n)) = (1+q^{-2})\left(\frac{1}{1-q^2}\right)^2$

suggests a new 2-morphism of degree -2:

$\deg\Big($ $\Big) = -2$

such that

$= 0.$

$\Big($ Also, $\Big)$

- $$\mathrm{gdim}\left(\mathrm{HOM}_{\mathcal{U}}(FE\mathbb{1}_n, \mathbb{1}_n)\right) = \langle FE\mathbb{1}_n, \mathbb{1}_n\rangle = \langle E\mathbb{1}_n, T(F)\mathbb{1}_n\rangle$$
$$= q^{1+n}\langle E\mathbb{1}_n, E\mathbb{1}_n\rangle = \frac{q^{1+n}}{1-q^2}$$

and similar calculations suggest new generators

I won't write down all the relations — see [Khovanov–Lauda]

- KLR algebras (a.k.a. nilHecke algebra, for sl_2)

Consider only braid-like diagrams (strands are all oriented upward)

This categorifies the positive half of U.

- KLRW algebra categorifies tensor product representations.
  (tensor product algebra)

KLRW algebra

$$V^{\underline{\lambda}} = V^{\lambda_1} \otimes \cdots \otimes V^{\lambda_\ell} \rightsquigarrow \mathcal{B}^{\underline{\lambda}} := T^{\underline{\lambda}}\text{-mod},$$

$$\mathcal{V}^{\underline{\lambda}} := D(\mathcal{B}^{\underline{\lambda}})$$

σ a braid $\rightsquigarrow$ $\mathbb{B}_\sigma : \mathcal{V}^{\underline{\lambda}} \longrightarrow \mathcal{V}^{\sigma(\underline{\lambda})}$

evaluation $\rightsquigarrow$ $\mathbb{E} : \mathcal{V}^{\underline{\lambda}} \longrightarrow \mathcal{V}^{\underline{\lambda}^-}$

coevaluation $\rightsquigarrow$ $\mathbb{K} : \mathcal{V}^{\underline{\lambda}^-} \longrightarrow \mathcal{V}^{\underline{\lambda}}$

For simplicity, let's just consider the case $\lambda_1 = \cdots = \lambda_\ell = 1$.

<u>Def</u> Let $\widetilde{T}^{\ell}_{n=\ell-2k}$ be the graded algebra spanned by braid-like diagrams with ℓ straight red strands (fixed) and k black strands, modulo local relations

(Webster's convention, all strands are oriented downward ($\mathbb{F}$), but the convention of left/right is flipped from Lauda's)

$\left(\begin{array}{l} \deg(\uparrow) = 2 \\ \deg(X) = -2 \\ \deg(\times) = 1 \end{array}\right)$

Let $\widetilde{T}^{\ell} := \bigoplus_n \widetilde{T}^{\ell}_n$, and let T^{ℓ} be the quotient of $\widetilde{T}^{\ell}$ by the 2-sided ideal generated by "violating diagrams" $= 0$.

(Implicitly, 0 on the very left, n on the very right region.)

left most

"double tensor product algebra"

Webster shows that T^{Δ} is Morita equivalent to a bigger algebra DT^{Δ} where the black strands are allowed to turn.

For any sequence $u = \uparrow\downarrow\cdots\downarrow$, let e_u be the idempotent in DT^{ℓ} which act by the identity on all diagrams which end in u on top left.

Then $\beta'_u := e_u \cdot DT^{\ell}$ is a $DT^{\ell}_m - DT^{\ell}_n$ bimodule

, and any 2-morphism $\phi : u \to v$ in $\mathcal{U}$ defines a map of bimodules $\beta'_\phi : \beta'_u \to \beta'_v$.

Let $f : T^{\ell} \to DT^{\ell}$ be the natural map, and let $e_- \in DT^{\ell}$ be the idempotent given by the image of the identity (i.e. it makes sure that all the black strands are oriented downward on the top and the bottom).

The Morita equivalence is induced by bimodules $e_- DT^{\ell}$ and $DT^{\ell} e_-$.

Under Morita equivalence, β'_u becomes $\beta_u := e_- \beta'_u e_-$.

It follows that

<u>Thm</u> There is a 2-representation of $\mathcal{U}$ on $\mathcal{B}^{\ell} := T^{\ell}\text{-mod}$, which sends

$$n \longmapsto \mathcal{B}^{\ell}_n$$
$$u \longmapsto \beta_u$$
$$\phi \longmapsto \beta_\phi$$

In particular, $\mathcal{F} \longmapsto$

$\mathcal{E} \longmapsto$

Webster also shows that

$$\bigoplus_n K_0(T_n^\ell\text{-mod}) \otimes_{\mathbb{Z}} \mathbb{C} \cong (\mathbb{C}^2)^{\otimes \ell}$$

$$[S_\lambda] \longmapsto s_\lambda$$

For λ a partition in $k \times (\ell - k)$, e.g. $\ell = 5, k = 2$, $\lambda = (1,2)$

$$\updownarrow$$

$$\begin{pmatrix}1\\0\end{pmatrix} \otimes \underline{\begin{pmatrix}0\\1\end{pmatrix}} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \underline{\begin{pmatrix}0\\1\end{pmatrix}} \otimes \begin{pmatrix}1\\0\end{pmatrix}$$

a basis vector of $(\mathbb{C}^2)^{\otimes 5}$,

where S_λ is the "cell module" of T^ℓ

which has basis

(which is a subset of the "cellular basis" of T_n^ℓ.)

The classes $[S_\lambda]$ give a free basis of $K_0(T^\ell\text{-mod})$.

- Braiding is given by T^ℓ-T^ℓ bimodules β_σ

Let $B_\sigma := \beta_\sigma \overset{\mathbb{L}}{\otimes} - \langle 1 \rangle : D^b(T^\ell\text{-gmod}) \to D^b(T^\ell\text{-gmod})$

This turns out to be an equivalence, with inverse $\mathbb{R}\mathrm{Hom}(\beta_\sigma, -) : D^b(T^\ell\text{-gmod}) \to D^b(T^\ell\text{-gmod})$

- Coevaluation is given by

$$\mathbb{K}_i := K_i \overset{\mathbb{L}}{\otimes} - : D^b(T^\ell\text{-gmod}) \to D^b(T^{\ell+2}\text{-gmod}),$$

where K_i is the $T^{\ell+2}$–T^ℓ bimodule spanned by diagrams with a single cup in the i-th position

modulo

- Evaluation is given by

$$\mathbb{E}_i := \mathrm{RHom}(K_i, -)\underbrace{\{-1\}[1]}_{\langle -1\rangle} \cong \dot{K}_i \overset{\mathbb{L}}{\otimes} - \underbrace{\{1\}[-1]}_{\langle 1\rangle} : D^b(T^{\ell+2}\text{-gmod}) \to D^b(T^\ell\text{-gmod})$$

mirror image (through the x-axis)

Note, $K \in T_0^2$-mod admits a nice projective resolution

which can be used to show $\mathrm{RHom}(K, K) \cong H^*(S^2)$ with the usual Frobenius structure.

$$\mathrm{RHom}(K,K) \cong K \overset{\mathbb{L}}{\otimes} K\langle 2\rangle$$

- Recall $[\![\times]\!] \cong \mathrm{Cone}([\![ )( ]\!] \to [\![\asymp]\!]\{1\})$.

This is a cone of $\underline{1} \to \mathbb{K}\mathbb{E}\langle 1\rangle$, which should be the unit η of the adjunction $(\mathbb{E}, \mathbb{K}\langle 1\rangle)$,

T^ℓ as T^ℓ–T^ℓ bimodule $\quad K \underset{T^{\ell-2}}{\otimes} \dot{K}$

which sends $1 \in T^\ell$ to the sum of all diagrams with no crossings and a single pair of (all vertical).

In fact, there is an exact triangle $\mathbb{B}_\sigma \to \underline{1} \xrightarrow{\eta} \mathbb{K}\mathbb{E}\langle 1\rangle \xrightarrow{(1)} \mathbb{B}_\sigma[1]$, which shows that, via 2-functor $\mathcal{BN} \xrightarrow{\Upsilon} \mathrm{Cat}$, $\ell \longmapsto D^b(T^\ell\text{-gmod})$, we get Khovanov homology for tangles.

In fact, that

is an exact sequence of T^{ℓ}-T^{ℓ} bimodules is a relatively simple calculation.