Chapter 2: Regularity of Minimizing Harmonic Maps

§2.1: The Case of $\dim(M) = 2$.

We now study existence and regularity of Minimizing Harmonic maps. Much of what we say holds in all dimensions. However, the main theorems will be proved only in $\dim = 2$.

Recall: $W^{1,2}(M, \mathbb{R}^L)$ consists of $\mathbb{R}^L$-valued functions with L^2 distributional derivatives. We define.

$$W^{1,2}(M,N) := \left\{ v : M \to \mathbb{R}^L \;\middle|\; \begin{array}{l} \|v\|^2_{W^{1,2}(M)} = \int_M \left(|v|^2 + e_g(v)\right) dv_g < +\infty \\ v(x) \in N \text{ for a.e. } x \in M \end{array} \right\}$$

<u>Note</u>: $W^{1,2}(M,N) \subseteq W^{1,2}(M,\mathbb{R}^L)$ is <u>closed</u> under sequentially weak convergence in $W^{1,2}(M,\mathbb{R}^L)$.

The main issue with these spaces is that $C^\infty(M,N)$ <u>May Not</u> be dense in $W^{1,2}(M,N)$.

<u>Example</u> (Schoen–Uhlenbeck)

$\varphi(x) = \dfrac{x}{|x|} \in W^{1,2}(B^3, S^2)$ can't be approximated by smooth maps from B^3 to S^2 in $W^{1,2}(B^3, S^2)$.

<u>pf</u>: <u>First</u>, observe that

$$\int_{B^3} \left|\nabla\left(\frac{x}{|x|}\right)\right|^2 d\mathrm{vol} = \int_0^1 \int_{S^2} \frac{1}{r^2}\left|\nabla_{S^2}\left(\frac{x}{|x|}\right)\right|^2 r^2 \, dH^2 \, dr$$

<u>Since</u> $\nabla \dfrac{x}{|x|}$ lies tangent $S^2(r)$ when $|x| = r$, and

$g_{euc} = dr^2 + r^2 g_{S^2}$ <u>so</u> $|\cdot|_{g_{euc}} (|x| = r) = \dfrac{1}{r^2} |\cdot|_{g_{S^2}}$.

<u>Now</u>

$$\nabla_{S^2}\left(\frac{x}{|x|}\right) = \nabla_{S^2} x = \left(\nabla_{\mathbb{R}^3} x\right)^{\mathrm{proj}_{S^2}} \cancel{+ ((\vec{e}_1, \vec{e}_2, \vec{e}_3)) \cdot (x_1 e_1}$$

Now $\nabla x_i = e_i$ and $T_{\vec{x}} S^2 = \vec{X}^{\perp}$ so

$$\nabla_{S^2} \vec{X} = \nabla_{\mathbb{R}^3} \vec{X}\Big|_{TS^2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then

$$|\nabla_{S^2} \vec{X}| = 2$$

Thus

$$\int_{B^3} \left|\nabla\left(\frac{x}{|x|}\right)\right|^2 = 2 \int_{S^2} dH^2$$

$$= 2\, H^2(S^2) = 8\pi.$$

Now, suppose $\exists\, u_i \in C^\infty(B^3, S^2)$ s.t. $\lim_{i\to\infty} \|\nabla(u_i - \varphi)\|_{L^2(B^3)} = 0$

Then by Fatou's Lemma and Fubini's Theorem

$$0 = \liminf_{i\to\infty} \int_0^1 \int_{\partial B_r} |\nabla(u_i - \varphi)|^2 \geq \int_{\frac{1}{2}}^{1} \liminf_{i\to\infty} \int_{\partial B_r} |\nabla(u_i - \varphi)|^2$$

So $\exists\, r \in (\frac{1}{2}, 1]$ s.t.

$$\liminf_{i\to\infty} \int_{\partial B_r} |\nabla(u_i - \varphi)|^2 = 0.$$

~~But~~ Recall: The topological degree (≡ winding #) of $f \in C^1(\partial B_r, S^2)$ is

$$\deg(f;\partial B_r) = \frac{1}{4\pi}\int_{\partial B_r} \det(\nabla_T f)\, d\mathcal{H}^2$$

where ∇_T is the Tagential gradient.

Then

① $\deg(\varphi;\partial B_r) = 1.$ since

$$\nabla_T \varphi = \nabla_T \frac{x}{|x|} = \frac{1}{r}\nabla_T \vec{x} = \frac{1}{r}\,\mathbb{1}_d\Big|_{TS^2} = \begin{bmatrix} \frac{1}{r} & 0 \\ 0 & \frac{1}{r} \end{bmatrix}$$

So $\det(\nabla_T \varphi) = \frac{1}{r^2} \Rightarrow \det(\varphi;\partial B_r) = 1.$

② $\deg(u_i;\partial B_r) = 0$ since, if u_i is smooth, then $\deg(u_i;\partial B_r)$ is a smooth function of r. But it is also integer valued. Thus $\deg(u_i;\partial B_r)$ is independent of r. Moreover

$$\deg(u_i;\partial B_r) \lesssim Cr^2 \longrightarrow 0 \text{ as } r \to 0$$

Thus $\deg(u_i;\partial B_r) = 0.$

But, since $\|\nabla(u_i-\varphi)\|_{L^2(\partial B_r)} \to 0$ $\nabla u_i \to \nabla\varphi$ pointwise a.e. and so

$$0 = \det(u_i;\partial B_r) = \frac{1}{4\pi}\int_{\partial B_r} \det(\nabla_T u_i)\,dH^2 \to \frac{1}{4\pi}\int_{\partial B_r} \det(\nabla_T \varphi)\,dH^2$$

$= 1$. This gives a contradiction. ∎

$\boxed{\text{The Idea}}$ L^2 convergence of gradients is enough to obtain equality of winding numbers. But smooth functions <u>all</u> have zero winding number.

<u>However</u>, we <u>Do</u> Have

<u>Thm</u> (rmk 2.1.2)

For $\dim(M)=2$, $C^\infty(M,N)$ <u>is</u> dense in $W^{1,2}(M,N)$.

For $n\geq 3$, if ~~$\pi_2(N)$~~ $\pi_2(N)=\{0\}$, then $C^\infty(B^n,N)$ is dense in $W^{1,2}(B^n,N)$.

Recall: Let $N \hookrightarrow \mathbb{R}^L$ be the isometric embedding guaranteed by The Nash Embedding theorem.

Let $A(y): T_yN \otimes T_yN \to (T_yN)^\perp$ be the second Fundamental Form.

Def'n 2.1.3: A map $u \in W^{1,2}(M,N)$ is a *weakly harmonic map* if it satisfies the Harmonic map eq'n *weakly*. ie

$$\int_M g^{\alpha\beta}\left(\left\langle \frac{\partial u}{\partial x^\alpha}, \frac{\partial \varphi}{\partial x^\beta}\right\rangle + A(u)\left(\frac{\partial u}{\partial x^\alpha}, \frac{\partial u}{\partial x^\beta}\right)\varphi\right) dv_g = 0$$

for *any* $\varphi \in C_0^\infty(M, \mathbb{R}^L)$.

Def'n 2.1.4 A map $u \in W^{1,2}(M,N)$ is a minimizing harmonic map if, for any $\Omega \subseteq M$

$$E(u,\Omega) := \frac{1}{2}\int_\Omega e(u)\,dv_g \leq E(v,\Omega)$$

for any $v \in W^{1,2}(\Omega, N)$ with $u|_{\partial\Omega} = v|_{\partial\Omega}$. (~~In the sense of Traces!~~)

(In the sense of Traces!).

Direct Computations prove

Prop 2.1.5: Any minimizing map $u \in W^{1,2}(M,N)$ is weakly harmonic.

pf for any $\Omega \subseteq M$, $\varphi \in C_0^1(M, \mathbb{R}^L)$ and $t \in \mathbb{R}$

$$E(u,\Omega) \leq E(\Pi_N(u+t\varphi), \Omega).$$ ~~This result~~ ~~fall~~ implies

$$\frac{d}{dt}\Big|_{t=0} E(\Pi_N(u+t\varphi), \Omega) = 0.$$

The result follows by direct computation.

Prop 2.1.6 (Existence of weak minimizers)

For $\partial M \neq \emptyset$, suppose that $\varphi \in W^{1,2}(M,N)$ is given. Then $\exists$ at least one minimizing harmonic map $u \in W^{1,2}(M,N)$ with $u|_{\partial M} = \varphi|_{\partial M}$.

pf This is the usual calculus of variations argument. We include it for completeness.

take $\mathcal{A} = \{u \in W^{1,2}(M,N) \mid u|_{\partial M} = \varphi|_{\partial M}\} \neq \emptyset$.

let $c = \inf_{\mathcal{A}} E(v) \geq 0$.

Since $A \neq \emptyset$, $c \leq E(\varphi) < +\infty$. Let $\{u_i\} \subseteq A$ be any minimizing sequence; i.e.

$$\lim_{i \to \infty} \frac{1}{2} \int_M e(u_i)\, dvol_g = c.$$

Then recall the Poincaré inequality.

Lemma: Let (M,g) be a compact n-dim'l Riemannian mfd w/ bdry. If $1 < p < n$, there exists a constant $C > 0$ s.t. for all $u \in W^{1,2}(M)$

$$\int_M |u|^p dv_g \leq C \left(\int_M |\nabla u|^p dv + \int_{\partial M} |u|^p ds \right)$$

Pf: (sketch) Suppose not. Then $\exists\, u_m$ s.t.

$$\int_M |u_m|^p \geq m \left(\int_M |\nabla u_m|^p dv + \int_{\partial M} |u_m|^p ds \right).$$

WLOG, assume $\int_M |u_m|^p dV = 1$. Then

$$\int_M |\nabla u_m|^p + \int_{\partial M} |u_m|^p \leq \frac{1}{m}. \Rightarrow \int_M |\nabla u_m|^p \to 0$$

and $\int_{\partial M} |u_m|^p \to 0$.

By weak ~~convergen~~ compactness $u_m \rightharpoonup u_\infty$.

<u>Moreover</u>, by the compactness of the embedding $W^{1,p}(M) \hookrightarrow L^p(M)$ (by Rellich's Lemma) $u_m \to u_\infty$ strongly in L^p. Hence $\|u_\infty\|_{L^p(M)} = 1. \Rightarrow u \neq 0$.

<u>But</u> by Fatou, we have

$$\int_M |\nabla u|^p \leq \liminf \int |\nabla u_m|^p = 0$$

<u>so</u> $\nabla u = 0$ <u>a.e</u>. so u is a.e. constant.

<u>but</u> since the trace map $W^{1,p}(M) \to L^p(\partial M)$ is continuous (<u>i.e</u> $\|Tu\|_{L^p(\partial M)} = \|u\|_{L^p(\partial M)} \leq C\|u\|_{W^{1,p}(M)}$) we have that $u\big|_{\partial M} = 0$ <u>a.e</u>. Thus $u = 0$ <u>a.e</u>. a contradiction. $\blacksquare$ <u>Lemma</u>.

As a result, since $u_i\big|_{\partial M} = \varphi\big|_{\partial M}$, we have

$$\|u_i\|_{W^{1,2}(M)} \leq C. \Rightarrow u_i \rightharpoonup u_\infty \text{ weakly.}$$

<u>Moreover</u>, By Rellich's Theorem

up to choosing a subsequence, $u_i \to u$ in $L^2(M)$ and $L^2(\partial M)$. ~~Also~~ Hence $u|_{\partial M} = \varphi|_{\partial M}$. $\Rightarrow u \in A$.

$E(u) \geq C$. But, by lower semicontinuity we have

$$E(u) \leq \liminf_{i \to \infty} \frac{1}{2}\int e(u_i) = C.$$

Thus, $E(u) = C$. $\blacksquare$

The assumption that $\partial M \neq \emptyset$ is not necessary. However, in these case there may be <u>No</u> non-trivial Harmonic maps.

<u>Example</u>: Consider $W: S^N \to T^k$ (sphere to Torus) we take round metric on S^N, flat metric on T^k. Then the Bochner Identity implies W is constant (see Prop 1.5.2 from last time).

<u>Theorem</u>: If $\dim(M) = 2$, and $u \in W^{1,2}(M,N)$ is a minimizing Harmonic Map, then $u \in C^\infty(M,N)$.

pf: we cover only the C^α estimate. The $C^{1,\alpha}$ estimate is similar. Note $C^{1,\alpha} \Rightarrow C^\infty$ by boot strapping.

The idea is to use the Morrey/Schauder Theory to compare a minimizing Harmonic map to a solution of Laplaces equation.

Since Reg. is local, we assume $(M,g) = (B_1, e^{-f} g_{euc})$. In fact, we can assume $f=0$, as will become clear from the proof. Fix $\varepsilon > 0$

1). Since $u \in W^{1,2}(M,N)$ $\exists\, r_0$ s $\forall\, r < r_0$, $x \in B_{1/2}$,

$$\int_{B_r(x)} |\nabla u|^2 \leq \varepsilon^2$$

$(\leq 8\varepsilon^2)$

Claim: $\exists\, r_1 \in (\frac{r}{2}, r)$ s.t. $r_1 \int_{\partial B_{r_1}(x)} |\nabla u|^2 \, dH^1 \leq 8 \int_{B_r / B(r/2)} |\nabla u|^2 dx$

pf if not, then

$$\int_{B_r / B_{r/2}} |\nabla u|^2 dx = \int_{r/2}^{r} \left(\int_{\partial B_s(x)} |\nabla u|^2 d H^1 \right) ds \geq \int_{r/2}^{r} \frac{8}{s} \left(\int_{B_r / B(r/2)} |\nabla u|^2 dx \right) ds$$

Thus $1 \geq 8\ln(2) > 1$ a contradiction.

Now $u \in W^{1,2}(\partial B_{r_1}(x))$ w/ $\int_{\partial B_{r_1}(x)} |\nabla u|^2 \leq \frac{8\varepsilon^2}{r_1}$.

Since ~~$\dim(\partial B_{r_1}(x)) = 1$, we can apply the Sobolev Imbedding Theorem to~~ get $u \in C^{1/2}(\partial B_{r_1}(x))$

Thus $\exists C$ s.t.

$$|u(z) - u(y)| \leq C\varepsilon \left(\frac{|y - z|}{r_1} \right)^{1/2} \quad \forall y, z \in \partial B_{r_1}(x)$$

In particular, if $p_0 = u(y)$, then

$$u(\partial B_{r_1}(x)) \subseteq B^L_{C\varepsilon}(p_0) \cap N$$

where $B^L_{C\varepsilon}(p_0)$ denotes the Ball in $\mathbb{R}^L$ $(N \hookrightarrow \mathbb{R}^L)$

Let v solve
$$\begin{cases} \Delta v = 0 \\ v\big|_{\partial B_{r_1}(x)} = u \end{cases} \qquad v : B_{r_1}(x) \to \mathbb{R}^L.$$

~~Let~~ Claim: $V(B_{r_1}(x)) \subseteq B^L_{C\varepsilon}(p_0)$.

pf

Consider $|V-p_0|^2$

Let $\Delta|V-p_0^2| = \sum_{k=1}^{2} \partial_k \sum_{i=1}^{L} 2(\partial V^i)(V^i-p_i) + 2(\partial_k V^i)^2$

since $\Delta V=0$ ($\Delta V^i=0$) we get

$\Delta|V-p_0|^2 \geq 0 \Rightarrow \max_{B_{r_1}(x)} |V-p_0|^2 \leq \max_{\partial B_{r_1}(x)} |V-p_0|^2$

But $V\Big|_{\partial B_r(x)} = u$ so $|V-p_0|^2 \leq |u-p_0|^2\Big|_{\partial B_{r_1}(x)} \leq C\varepsilon.$

$\blacksquare$ claim.

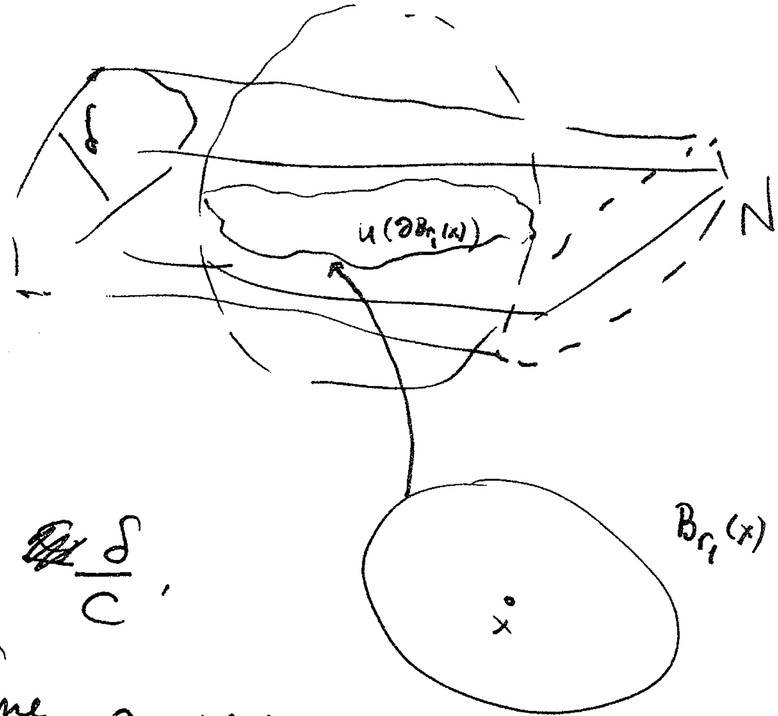
~~Let Π_N denote the orthogonal projection from~~

Let $\Pi_N : \mathbb{R}^L \to N$ be the map that sends x to $y \in N$ s.t. $dist(x,y) = dist(x,N)$

Let N_δ denote a tubular Nbhd of N of radius δ.

For δ sufficiently small

$\Pi_N : N_\delta \to N$ is C^∞.



Then for ~~ε suff~~ $\varepsilon < \frac{\delta}{C}$,

$B^L_{C\varepsilon}(p_0) \subseteq N^\delta$. Define a new map

$W : B_{r_1}(x) \to N$ by $W(y) = \Pi_N(V(y))$.

~~Then, since $u : B_{r_1}(x) \to N$ is minimizing~~

<u>Claim</u> :
$$\int_{B_{r_1}(x)} |\nabla u|^2 \leq \int_{B_{r_1}(x)} |\nabla \Pi_N(V)|^2 \qquad (*)$$

<u>pf</u> Note $W = u$ on $\partial B_{r_1}(x)$. Thus if $(*)$ did not hold, we could define

$\hat{u} : M \to N$ by
$$\hat{u}(x) = \begin{cases} u & \text{if } x \in M \setminus B_{r_1} \\ W & \text{if } x \in B_{r_1} \end{cases}$$

This is still $W^{1,2}$, and has energy strictly smaller than the energy of u. But u is minimizing, so we get a contradiction. $\boxed{\text{IN}}$ claim.

Thus

$$\int_{B_{r_1}(x)} |\nabla u|^2 \leq \int_{B_{r_1}(x)} |\nabla \Pi_N(v)|^2 \leq C \int_{B_{r_1}(x)} |\nabla v|^2.$$

Claim: if $\begin{cases} \Delta v = 0 & \text{on } B_{r_1}(x) \\ v = u & \partial B_{r_1}(x) \end{cases}$ then $\exists\, C > 0$ s.t.

$$\int_{B_{r_1}(x)} |\nabla v|^2 \leq C r_1 \int_{\partial B_{r_1}(x)} |\nabla u|^2.$$

pf assume $r_1 = B_1$, $x = 0$. General case follows from scaling. Assume also that $v(0) = 0$ (else traslate).

Then $$\int_{B_1} |\nabla v|^2 = \int_{B_1} \nabla \cdot (v \nabla v) = \int_{B_1} v \nabla v \cdot \vec{\eta} \leq \varepsilon \int_{\partial B_1} v^2 + \frac{1}{\varepsilon} \int_{\partial B_1} |\nabla v|^2$$

Thus, it suffices to show

$$\int_{\partial B_1} V^2 \le C \int_{B_1} |\nabla V|^2.$$ This is easy

$x \in \partial B_1$ then $V(x) = \overset{0}{\overset{\shortparallel}{V(0)}} + \int_{\ell} \nabla V \cdot \ell \, dt$ where ℓ joins zero to x. Then

$$V(x)^2 \lesssim \cancel{V(0)^2} + C \int_0^x |\nabla V|^2(\ell(x))\, dt$$

$$\Rightarrow \int_{\partial B_1} V(x)^2 \le C \int_{B_1} |\nabla V|^2 dx.$$ The claim follows. $\blacksquare$

<u>Question</u>: Is there an easier way to see this?

<u>Now</u>, we obtain

$$\int_{B_{r_1}(x)} |\nabla u|^2 \le C \int_{B_{r_1}(x)} |\nabla V|^2 \le C r_1 \int_{\partial B_{r_1}(x)} |\nabla u|^2 \le C \int_{B_r / B_{(r/2)}} |\nabla u|^2$$

$$\begin{array}{l} \vee\!/ \\ \displaystyle\int_{B_{r/2}} |\nabla u|^2 \end{array}$$

$$\Rightarrow (C+1) \int_{B_{r/2}} |\nabla u|^2 \le C \int_{B_r} |\nabla u|^2.$$

So $\displaystyle\int_{B_{r/2}} |\nabla u|^2 \leq \Theta \int_{B_r} |\nabla u|^2 \qquad \Theta = \frac{c}{c+1} < 1.$

Then

$$\int_{B_{2^{-k}r}} |\nabla u|^2 \leq \Theta^k \int_{B_r} |\nabla u|^2 \quad \forall x \in B_{1/2} \text{ and } 0 < r \leq r_0.$$

Set $\alpha = \dfrac{|\ln \theta|}{2 \ln 2} \in (0,1)$, then

$$\int_{B_r(x)} |\nabla u|^2 \leq C \left(\frac{r}{r_0}\right)^{2\alpha} \int_{B_{r_0}(x)} |\nabla u|^2 \quad \forall x \in B_{\frac{1}{2}} \quad 0 < r \leq r_0.$$

Lemma: For $\alpha \in (0,1)$ and any open $B \subseteq \mathbb{R}^n$, if

~~B.t~~ $f: B \to \mathbb{R}$ satisfies

$$\sup_{B_r(x) \subseteq B} \left(r^{2(1+\alpha)-n} \int_{B_r(x)} |\nabla f|^2 \right) < +\infty$$

then $u \in C^\alpha(B)$.

(pf: see Fanghua Lin)