Harmonic Maps & their heat flows

Chapter 1: Introduction.

Def'n: Let $(M^n, g_{\alpha\beta})$ be an n-dim'l Riemannian mfd, and (N^l, h_{ij}) be an l-dim'l Rmn mfd. We assume N has no boundary, but allow M to have bdry. Let $u \in C^2(M,N)$. Then, at a point $p \in M$, we define

$$e(u)(p) \equiv |\nabla u|_g^2 = \frac{1}{2} g^{\alpha\beta}(p)\, h_{ij}(u(p)) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta}.$$

Then the ~~dr~~ Dirichlet energy functional is

$$E(u) = \int_M e(u)\, dVol_g.$$

Examples: Let $M = \Omega \subseteq \mathbb{R}^n$ be a domain, and let $N = \mathbb{R}$. Both mfds equipped with the Euclidean metric. Then

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dV,$$

~~and~~ is the usual Dirichlet Energy.

__Def'n__: A map $u \in C^2(M,N)$ is a __Harmonic Map__ if it is a critical point of ~~the~~ ~~Ear~~ E.

__Example__: In the previous example, a harmonic map $u \in C^2(\Omega, \mathbb{R})$ is a Harmonic Function.

__Proposition__ 1.1.2 (Functional Eqn).

A map $u \in C^2(M,N)$ is a harmonic map if and only if

$$\boxed{\Delta_g u^i + g^{\alpha\beta}\Gamma^i_{jk}\frac{\partial u^j}{\partial x^\alpha}\frac{\partial u^k}{\partial x^\beta} = 0}$$

where $\Delta_g = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{g}\, g^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\right)$ is the Laplace-Beltrami operator and Γ^i_{jk} are the Christoffel symbols of the metric h on N.

__proof__: Fix $U \subseteq M$ any coordinate chart, and let $\varphi \in C^2_0(U, \mathbb{R}^\ell)$. Then

$$0 = \frac{d}{dt}\bigg|_{t=0}\left(\frac{1}{2}\int_M g^{\alpha\beta} h_{ij}(u+t\varphi)(u^i_\alpha + t\varphi^i_\alpha)(u^j_\beta + t\varphi^j_\beta)\right.$$

(1)

$$= \frac{1}{2} \int_M g^{\alpha\beta} h_{ij,k} \varphi^k u^i_\alpha u^j_\beta \sqrt{g}\, dv.$$

$$+ \int_M g^{\alpha\beta} h_{ij} u^i_\alpha \varphi^j_\beta \sqrt{g}\, dv.$$

Now, we integrate by parts in the second term and get.

$$0 = (1) - \int_M g^{\alpha\beta} h_{ij,l} u^l_\beta u^i_\alpha \varphi^j \sqrt{g}\, dV$$

$$- \int_M \Delta_g u^i h_{ij} \varphi^j \sqrt{g}\, dv.$$

Thus

$$\int_M \Delta_g u^i h_{ij} \varphi^j \sqrt{g}\, dv = \int_M g^{\alpha\beta} \left(\frac{1}{2} h_{ij,k} - h_{ik,j} \right) \varphi^k u^i_\alpha u^j_\beta\, dv$$

But $$g^{\alpha\beta} h_{ik,j} \varphi^k u^i_\alpha u^j_\beta = \frac{1}{2} g^{\alpha\beta} \left(h_{ik,j} + h_{jk,i} \right) \varphi^k u^i_\alpha u^j_\beta$$

So

$$\int_M \Delta_g u^i h_{ij} \varphi^j \sqrt{g}\, dv = \int_M \frac{1}{2} g^{\alpha\beta} \left(h_{ij,k} - h_{ik,j} - h_{jk,i} \right) \varphi^k u^i_\alpha u^j_\beta$$

Take $\varphi^j = h^{jm} \eta_m$ for $\eta \in C_0^2(U, \mathbb{R}^l)$. Then

$$\int_M \Delta_g u^i \eta^i = \int g^{\alpha\beta} \Gamma^i_{jk} u^j_\alpha u^k_\beta \eta_i \, dV$$

so the propositon is proved. $\blacksquare$

1.2: Intrinsic View of Harmonic Maps.

$u \in C^2(M,N)$, T^*M the cotangent bundle and u^*TN the pull-back of TN by u. We view

$du := \frac{\partial u}{\partial x^\alpha} dx^\alpha \otimes \frac{\partial}{\partial u^i}$ as a section of $T^*M \otimes u^*TN$.

Then $e(u) = \frac{1}{2}\langle du, du\rangle = \frac{1}{2} Tr_g(u^*h)$ where

$(u^*h)_{\alpha\beta} = h_{ij}(u) u^i_\alpha u^j_\beta$. Let ∇ be the covariant derivative on $T^*M \otimes u^*TN$. Then.

Proposition 1.2.1 $u \in C^2(M,N)$ is Harmonic iff

$$\tau(u) := Tr_g(\nabla du) = 0 \quad \text{in } M.$$

We will not prove this result. However, we can compute locally this equation:

$$\nabla du = \nabla_\gamma\left(u^i_\alpha\, dx^\alpha \otimes \frac{\partial}{\partial u^i}\right)$$

$$= u^i_{\alpha\gamma}\, dx^\alpha \otimes \frac{\partial}{\partial u^i} - (\Gamma^M)^\alpha_{\gamma\beta}\, u^i_\alpha\, dx^\beta \otimes \frac{\partial}{\partial u^i}$$

$$+ (\Gamma^N)^m_{li}\, u^l_\gamma u^i_\alpha\, dx^\alpha \otimes \frac{\partial}{\partial u^m}$$

So

$$\boxed{\tau^k(u) = g^{\alpha\gamma}\left(u^i_{\alpha\gamma} - u^i_\beta \Gamma^\beta_{\gamma\alpha} + (\Gamma^N)^i_{lm}\, u^l_\gamma u^m_\alpha\right) = 0}$$

In Local coordinates, this is equivalent to

$$\tau^k(u) = g^{\alpha\beta}\left(u^k_{\alpha\beta} - (\Gamma^M)^\gamma_{\alpha\beta}\, u^k_\gamma + (\Gamma^N)^k_{ij}(u)\, u^i_\alpha u^j_\beta\right) = 0$$

1.3: Extrinsic View of Harmonic Maps:

Assume $(N,h) \overset{\text{isom}}{\hookrightarrow} (\mathbb{R}^L, g_{euc})$. Then

$$C^2(M,N) = \{\vec{u} = (u^1, \ldots, u^L) \in C^2(M,\mathbb{R}^L) \mid u(M) \subseteq N\}.$$

Then

$$e(u) = \frac{1}{2} g^{\alpha\beta} u^i_\alpha u^i_\beta.$$

$\exists\, \delta = \delta(N) > 0$ s.t. the map $\Pi_N : N_\delta \to N$ is smooth, where

$$N_\delta = \{y \in \mathbb{R}^L \mid d(y,N) < \delta\}.$$

and

$$\Pi_N(y) = \tilde{y} \in N \text{ s.t. } d(y,\tilde{y}) = d(y,N).$$

is the "nearest point map". Then

(1). $P(y) = \nabla \Pi_N(y) : \mathbb{R}^L \to T_yN$, $y \in N$ is an orthogonal projection map pf (i) since we are in $\mathbb{R}^L$, ∇ is just the directional derivative. Fix $\vec{v} \in \mathbb{R}^L = T_y\mathbb{R}^L \supseteq T_yN$.

~~Then~~ Let $\gamma(t)$ be a ~~geodesic~~ curve through y w/ tangent $\vec{v}$. Then

$$\Pi_N(\gamma(t)) = \begin{cases} y & \text{if } v \in (T_yN)^\perp. \\ \gamma(t) & \text{if } v \in T_yN \end{cases} \quad \left(\begin{array}{c}\text{Take } \gamma(t) \\ \text{to be a} \\ \text{geo. in } N\end{array}\right)$$

So

$$\nabla_v \Pi_N(y) = \frac{d}{dt}\Big|_{t=0} \Pi_N(\gamma(t)) = \begin{cases} 0 & \text{if } v \in (T_yN)^\perp \\ v & \text{if } v \in T_yN. \end{cases}$$

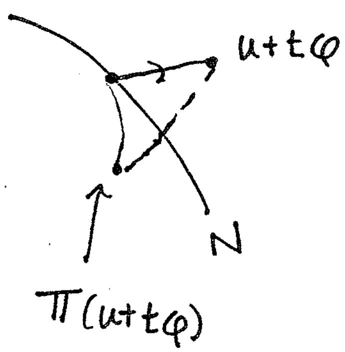
(ii) ~~$\nabla\Pi_N(y)$~~ The map

$$A(y) = \nabla P(y) : T_yN \otimes T_yN \longrightarrow (T_yN)^\perp$$

is the second Fundamental form of $N \subseteq \mathbb{R}^L$.

Proposition 1.3.1: $u \in C^2(M,N)$ is a harmonic map iff u satisfies $\Delta u_g \perp T_uN$.

pf: Fix $\varphi \in C^2(M,\mathbb{R}^\ell)$ then $u + t\varphi$ defines a deformation of N.



So $\Pi(u+t\varphi) : M \to N$.

Then $E(\Pi(u+t\varphi)) = \int_M |\nabla \Pi(u+t\varphi)|^2 \, dvol$

So if $u \in C^2(M,N)$ is a critical pt., then

$$0 = \frac{d}{dt}\Big|_{t=0} \int_M |\nabla \Pi(u+t\varphi)|^2 \, dv = $$

Now $\frac{d}{dt}\Big|_{t=0} \langle \nabla \Pi(u+t\varphi), \nabla \Pi(u+t\varphi) \rangle$

$= 2 \langle \nabla \nabla_\varphi \Pi(u), \nabla \Pi(u) \rangle$. But $\Pi(u) = u$, so

$$0 = 2 \int \langle \nabla P(u)(\varphi), \nabla u \rangle dv = -2 \int \langle P(u)(\varphi), \Delta u \rangle dv$$

$$= -2 \int \langle \varphi, P(u) \Delta u \rangle \, dv$$

So $P(u)\Delta u = 0 \Rightarrow \Delta u \in (T_y N)^\perp$. ∎

$\boxed{\text{Example}}$: Let $M = T^n$, $N = S^k \subseteq \mathbb{R}^{k+1}$.

Then a map $u \in C^2(T^n, S^k)$ is harmonic if and only if

$$\Delta u + |\nabla u|^2 u = 0.$$

Pf. Let $u : T^n \to S^k$. $\vec{u} = (u_1, \dots, u_{k+1})$. Then $u_1^2 + \dots + u_{k+1}^2 = 1$ so for y_α $1 \leq \alpha \leq n$ we have

$$\sum_{i=1}^{k+1} \left(\frac{\partial u_i}{\partial y_\alpha}\right)^2 + u_i \frac{\partial^2 u_i}{\partial y_\alpha^2} = 0.$$ Summing over α

gives:

$$|\nabla u|^2 + \vec{u} \cdot \Delta \vec{u} = 0.$$

Now, if $\vec{u}$ is harmonic, then $\Delta \vec{u} = c\vec{u}$ for some c.

So $\vec{u} \cdot \Delta \vec{u} = c\, \vec{u} \cdot \vec{u} = c$. But $\vec{u} \cdot \Delta \vec{u} = -|\nabla u|^2$

Hence

$$\Delta \vec{u} = -|\nabla u|^2 \vec{u}$$

∎

§ 1.4 : A few facts about Harmonic Maps.

(1) If $\Phi : M \to M$ is a C^2-diffeo, and $u \in C^2(M,N)$ is a Harmonic map on (M,g), then $u \circ \Phi$ is a harm. map wrt. $(M, \Phi^* g)$.

(2) Let (M,g_1) be a Riemann Surface, then if $\Phi : (M,g_1) \to (M,g_2)$ is a conformal map, and $u \in C^2(M,N)$ is harmonic wrt. (M,g_2), then $u \circ \Phi$ is harmonic wrt (M, g_1).

(3) Harmonic maps from $S^1 \to N$ correspond to closed geodesics in N.

Pf Let $u : S^1 \to N$, then u is Harmonic iff

$$\frac{d^2u^i}{dt^2} + \Gamma^i_{jk}\frac{du^j}{dt}\frac{du^k}{dt} = 0 \iff u \text{ is a geodesic.}$$

(4) Harmonic maps from a Riemann surface depend only on the conformal structure.

(5) $Id : (M,g) \to (M,g)$ is harmonic.

(6) $\dim(M) = 2$, then a conformal map is harmonic.

§1.5: The Bochner Identity for Harmonic Maps

Let $u \in C^2(M,N)$. Let

$R^M = R_{\alpha\beta\gamma\delta}$, $Ric^M = R_{\alpha\beta}$ $\quad R^N = \hat{R}_{ijkl}$.

$\boxed{\text{Thm 1.5.1}}$

If $u \in C^2(M,N)$ is a harmonic map, then

$$\Delta_g e(u) = |\nabla du|^2 + R_{\alpha\beta}(u_\alpha, u_\beta) - \hat{R}_{ijkl}(u)(u^i_\alpha, u^j_\beta, u^k_\alpha, u^l_\beta)$$

where ∇ denotes the covariant derivative on $T^*M \otimes u^*TN$.

<u>pf</u> We just compute. Fix $p \in M$, and normal coords near p, $u(p) \in N$. Then

$$\Delta e(u) = \frac{1}{2} \partial_\gamma \partial_\gamma \left(g^{\alpha\beta} h_{ij}(u) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} \right)$$

$$= \overset{①}{\frac{1}{2} \left(\partial_\gamma \partial_\gamma g^{\alpha\beta} \right) u^i_\alpha u^i_\beta} + \overset{②}{\frac{1}{2} \left(\partial_l \partial_k h_{ij} \right) u^l_\gamma u^k_\gamma u^i_\alpha u^j_\alpha}$$

$$+ \overset{③}{\cancel{\tfrac{1}{2}} \left(u^i_{\alpha\gamma\gamma} u^i_\alpha + u^i_{\alpha\gamma} u^i_{\alpha\gamma} \right)}$$

Recall that, in Normal coordinates, the metric expands as

$$g_{\alpha\beta} = \delta_{\alpha\beta} - \frac{1}{3} R_{\alpha\gamma\beta\delta} X^\gamma X^\delta + O(|x|^3)$$

$$h_{ij} = \delta_{ij} - \frac{1}{3} \hat{R}_{ikj\ell} X^k X^\ell + O(|x|^3)$$

So:

$$1: \quad \partial_\gamma \partial_\gamma g^{\alpha\beta} = -g^{\alpha\nu} \partial_\gamma \partial_\gamma g_{\nu\eta} g^{\eta\beta} = -\partial_\gamma \partial_\gamma g_{\alpha\beta}$$

$$= \frac{2}{3} R_{\alpha\gamma\beta\gamma}$$

So $1 = \frac{1}{3} Ric^M(\nabla u^i, \nabla u^i)$

Similarly

$$2 = -\frac{2}{3} \hat{R}(u_\alpha, u_\gamma, u_\alpha, u_\gamma)$$

Finally, we simplify 3. By Prop 1.2.1 we have

$$\partial_\alpha(\Delta u^i) = + \partial_\alpha\left(u^i_\beta (\Gamma^M)^\beta_{\gamma\gamma} - (\Gamma^N)^i_{\ell m} u^\ell_\gamma u^m_\gamma \right)$$

& Now, $\Gamma^M(p) = \Gamma^N(u(p)) = 0$

So we get

$$\partial_\alpha(\Delta u^i) = u^i_\beta\, \partial_\alpha (\Gamma^M)^\beta_{\gamma\gamma} - \partial_\alpha (\Gamma^N)^i_{lm}\, u^l_\gamma u^m_\gamma .$$

$$\partial_\alpha \Gamma^{M\,\beta}_{\gamma\gamma} = \partial_\alpha\left(\frac{1}{2} g^{\beta\eta}\left(2\partial_\gamma g_{\eta\gamma} - \partial_\eta g_{\gamma\gamma}\right)\right)$$

$$= \frac{1}{2}\left(2\partial_\alpha\partial_\gamma g_{\beta\gamma} - \partial_\alpha\partial_\beta g_{\gamma\gamma}\right)$$

$$= -\frac{\cancel{2}\,1}{3} R_{\beta\alpha\gamma\gamma} + \frac{1}{\cancel{3}} R_{\gamma\alpha\gamma\beta} + \frac{1}{6} R_{\gamma\beta\gamma\alpha} - \frac{1}{3} R_{\beta\gamma\gamma\alpha} \cancel{-\tfrac{1}{2}}$$

$$= \frac{2}{3} R_{\alpha\gamma\beta\gamma}$$

~~Similarly, we obtain~~ Carrying out a similar computation ~~$\partial_\alpha((\Gamma^N)^i_{lm})$~~ for $\partial_\alpha \Gamma^N$, we obtain.

$$\partial_\alpha(\Delta u^i)\, u^i_\alpha = \frac{2}{3} R_{\alpha\gamma\beta\gamma}\, u^i_\alpha u^i_\beta - \frac{2}{3}\hat{R}(u_\alpha, u_\gamma, u_\alpha, u_\gamma)$$

So Finally, we have

$$\Delta e(u) = |u^i_{\alpha\beta}|^2 + \mathrm{Ric}^M(\nabla u^i, \nabla u^i) - \hat{R}(u_\alpha, u_\gamma, u_\alpha, u_\gamma)$$

It is straight forward to check that, at $p \in M$ we have $|u^i_{\alpha\beta}|^2 = |\nabla du|^2$. Thus, we have proved the Theorem. $\blacksquare$

As a Corollary, we obtain

$\boxed{\text{Prop 1.5.2}}$: Let (M,g) be ~~closed~~ a compact mfd w/o boundary, and with $Ric^M \geq 0$. Suppose that $K^N \leq 0$. Then, any harmonic map $u \in C^2(M,N)$ is totally geodesic. If $Ric^M > 0$ at a point of M, then u is constant. If $K^N < 0$, then either u is constant, or $u(M)$ is contained in a closed geodesic.

pf By the Bochner formula $\Delta e(u) \geq 0$.

Thus, $e(u) = C$, $|\nabla du|^2 = 0$.

we claim this implies u is totally geodesic. Recall

Def'n: $u: M \to N$ is a totally geodesic immersion if for any geodesic γ in M, $u(\gamma)$ is a geodesic in N.

by direct computation, we have

$$\frac{d\,u(\gamma)^i}{dt} = \frac{\partial u^i}{\partial x^\alpha}\dot{x}^\alpha$$

$$\frac{d^2 u(\gamma)^i}{dt^2} = \frac{\partial^2 u^i}{\partial x^\alpha \partial x^\beta}\dot{x}^\alpha \dot{x}^\beta + \frac{\partial u^i}{\partial x^\alpha}\ddot{x}^\alpha$$

Since u has $|\nabla du|^2 = 0$, we know $\nabla du = 0$, and so, by the computation in the proof of ~~Thm~~ Prop 1.21. we have

$$u^i_{\alpha\beta} = u^i_\gamma (\Gamma^M)^\gamma_{\alpha\beta} - (\Gamma^N)^i_{lm} u^l_\alpha u^m_\beta$$

Thus

$$\frac{d^2 u(\gamma)^i}{dt^2} + (\Gamma^N)^i_{lm} \frac{du(\gamma)^l}{dt}\frac{du(\gamma)^m}{dt}$$

$$= u^i_\alpha\left(\ddot{x}^\alpha + \dot{x}^\beta \dot{x}^\gamma (\Gamma^M)^\alpha_{\beta\gamma}\right) - (\Gamma^N)^i_{lm} u^l_\alpha u^m_\beta \dot{x}_\alpha \dot{x}_\beta$$

$$+ (\Gamma^N)^i_{lm} u^l_\alpha u^m_\beta \dot{x}_\alpha \dot{x}_\beta$$

$$= 0.$$

Thus, as a consequence $u(\gamma)$ is geodesic.

Note: if $Ric^M > 0$, then $\nabla u^i_\alpha = 0$ $\forall i$ and so ~~it is constant~~. at P_0 $|\nabla u|_{(P_0)} = 0$. Thus $e(u) = 0$, and so $|\nabla u| = 0$ everywhere.

Finally, if $K^N < 0$, then either $u_\alpha = 0$ $\forall \alpha$ in which case u is constant, or the span $\{u_\alpha\}$ has dimension 1. In this case $u(M)$ is contained in a closed geodesic. ∎

§1.6: The second variational Formula for Harmonic maps.

Prop 1.6.2: Let $u \in C^2(M,N)$ be a Harmonic map, and $u_t \in C^2(M, N \times [0,1])$ be a family of smooth variations ($u_0 = u$). Let $V = \frac{du_t}{dt}\Big|_{t=0} \in C^2(M, u^*TN)$

Then.

$$\frac{d^2}{dt^2}\Big|_{t=0} E(u_t) = \int_M |\nabla V|_g^2 - tr_g \langle R^N(V, \nabla u)V, \nabla u\rangle dg.$$

In particular, if $K^N \leq 0$, then u is stable.

pf: just compute. Similar to the proof of prop 1.1.2.

Example: if $u \in C^2(M, S^k)$ is a Harmonic map, and $\varphi \in C_0^2(M, \mathbb{R}^{k+1})$, then

$$\frac{d^2}{dt^2} E\left(\frac{u+t\varphi}{|u+t\varphi|}\right) = \int_M \left(|\nabla \hat{\varphi}|^2 - |\nabla u|^2 |\hat{\varphi}|^2\right) dv$$

where $\hat{\varphi} := \varphi - \langle u, \varphi \rangle u$.