Perelman's Bounds on Scalar Curvature and Ricci Potential along the KRF.

~~Outline~~

Thm (Perelman)
Let $g(t)$ be a KRF on a compact, Kähler manifold M, $\dim_{\mathbb{C}} M = n$, with $c_1(M) > 0$. There exists a uniform constant C depending only on the initial metric so that for the Ricci potential u we have

$$|R(g(t))| + \operatorname{diam}(M, g(t)) + |u|_{C^1} \leq C$$

where the C^1-norm is wrt the evolving metric $g(t)$ and u is normalized so that $(2\pi)^{-n} \int_M e^{-u} dV_{g(t)} = 1$.

Outline of the proof.

(1) Uniform Lower bound on u.

(2) use maximum Principle to bound $|\nabla u|$, R by u

(3) steps 1, 2 $\Rightarrow$ $\sqrt{u + 2B}$ is uniformly Lipschitz, where B is ~~the~~ a constant s.t. $u + 2B \geq 0$, as guaranteed by step 1. This reduces the problem to a diameter bound

(4) use non-collapsing and the W functional to obtain the diameter bound.

we assume $c_1(M) = [\omega_0]$.

~~(1) Uniform Lower~~

<u>set up</u> : $g_{\bar{k}j}(t)$ solves $\begin{cases} \dfrac{\partial g_{\bar{k}j}}{\partial t} = -R_{\bar{k}j} + g_{\bar{k}j} = \partial_j \partial_{\bar{k}} u \\ g_{\bar{k}j}(0) = (g_0)_{\bar{k}j} . \end{cases}$

we normalize $u(t)$ by the condition $\int_M e^{-u} = (4\pi)^n$.

The Kähler potential $\varphi(t)$, defined by

$g_{\bar{k}j}(t) = (g_0)_{\bar{k}j} + \partial_j \partial_{\bar{k}} \varphi$ satisfies $\partial_j \partial_{\bar{k}} \dot{\varphi} = \partial_j \partial_{\bar{k}} u$.

Thus, we may take $\dot{\varphi} = u$. Recall, Last time we showed that $\mu(g(t), 1)$ was increasing along the Normalized KRF. <u>where</u>

$$\mu(g(t), \tau) = \inf_{\{f \mid \int_M (4\pi\tau)^{-n} e^{-f} dVol = 1\}} W(g(t), f, \tau).$$

we also showed $A := \mu(g(0), 1) > -\infty$.

<u>Lemma</u>: $\exists\ C = C_1(A)$ s.t. $\int u e^{-u} \geq C_1$.

<u>pf</u>

$$A = \mu(g(0), 1) \leq \int_M (4\pi)^{-n} e^{-u} (R + |\nabla u|^2 + u - 2n)\, dvol$$

$$= \int_M (4\pi)^{-n} e^{-u} (-\Delta u + |\nabla u|^2 + u - n)\, dvol$$

$$(4\pi)^{-n}\int \Delta(e^{-u}) + (4\pi)^{-n}\int u e^{-u} - n\int (4\pi)^{-n} e^{-u}\, dvol$$
$$= (4\pi)^{-n}\int u e^{-u} - n.$$ Hence Proved ☒

~~Lem~~ Remark: Set $a = (4\pi)^{-n}\int u e^{-u}$. Then a is monotonically increasing along the KRF. To prove this, observe that $\partial_j \partial_{\bar{k}} u = g_{\bar{k}j} - R_{\bar{k}j}$ and compute that u evolves by $\dot{u} = \Delta u + u - a$. Now compute evolution equation for a, and prove a Poincaré type Lemma.

$$\frac{1}{V}\int_M f^2 e^{-u}\, dvol \le \frac{1}{V}\int_M |\nabla f|^2 e^{-u}\, dvol + \left(\frac{1}{V}\int_M f e^{-u}\, dV\right)^2$$

Lemma: $\exists\, C_2$ uniform $C_2 > 0$ s.t. $a = (4\pi)^{-n}\int u e^{-u} \le C_2$.

pf write $u = u^+ - u^-$

$$a = (4\pi)^{-n}\int_M u e^{-u}\, dV = (4\pi)^{-n}\left[\int_M -u^- e^{u^-}\, dV + \int_M u^+ e^{-u^+}\, d\mu\right]$$

$$\le (4\pi)^{-n}\int_M u^+ e^{-u^+}\, dV \le C$$ since xe^{-x} is uniformly bounded on $\mathbb{R}^+$.

Lemma: The Scalar Curvature is uniformly bdd below

pf $$\dot{R} = \frac{d}{dt}\left(-g^{\bar{k}j}\partial_j\partial_{\bar{k}}\log\det g(t)\right) = \left\{ +g^{\bar{k}\ell}(g_{\bar{p}\ell} - R_{\bar{p}\ell})\, g^{\bar{p}q}(-R_{\bar{k}j}) \atop -g^{\bar{k}j}\partial_j\partial_{\bar{k}}(g^{p\bar{q}}\dot{g}_{p\bar{q}}) \right.$$

$$= -R + |Ric|^2 + \Delta R.$$

Now apply minimum principle.

Lemma: The Function $u(t)$ is uniformly bounded below.

Pf. Suppose $u << 0$ for some time t_0. Then, by the evolution equation for u, we have

$$(*) \quad \boxed{\dot{u} = \Delta u + u - a = n - R + u + a \leq C_1 + u < 0}$$

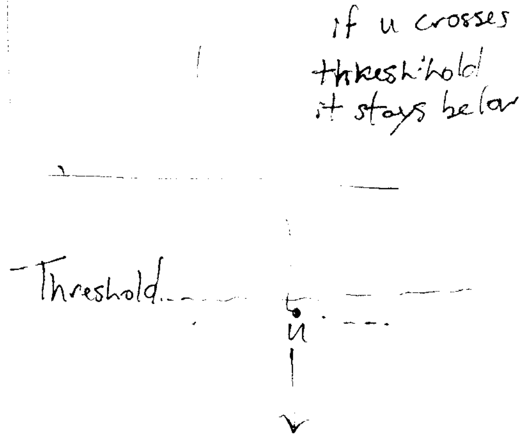
at $t = t_0$, and so $u(t)$ stays very negative for $t \geq t_0$.

If for some y_0, $u(y_0) << 0$ at some time t_0, then $\exists\, U$ open s.t. $u(y) << 0$ for all $y \in U$. By the above computation $u(y)$ ~~<<~~ stays very negative on U. $\forall t \geq t_0$. Then,

For $z \in U$.
integrate.
$$\frac{du}{dt} \leq c + u \Rightarrow u(t)(z) \leq e^{t-t_0}(c + u(t_0)) \leq -\tilde{C}_1 e^{t}$$

Now, $\dot{\varphi} = u \Rightarrow \varphi(t)(z) \leq \varphi(t_0)(z) - Ce^{t-t_0} \leq -c_1 e^{t}$ for t sufficiently large. But $(4\pi)^{-n}\int e^{-u}\,dV = 1$, so that $u(t)$ cannot be very ~~large~~ negative everywhere on M. ie $\exists\, C_2 \geq 0$ ~~s.t.~~ uniform s.t. $\max_M u(t) \geq -C_2$.

Picture



Now $\dot{\varphi} = u$, along with $(*)$ implies

$$\frac{d}{dt}(u - \varphi) = n - R + a \leq c_1$$

$$\Rightarrow (u-\varphi)(t) \leq ct + \max_M (u(0) - \varphi(0))$$

Let x_t' be s.t. $u(x_t', t) = \max_M u(t)$

For any point $x \in M$. ~~taking~~ max$_M$ on both sides yields then we get

$$u(x_t', t) - \varphi(x_t', t) \leq \max_M (u(0) - \varphi(0)) + ct$$

$$\Rightarrow \boxed{\max_M \varphi(t) \geq \varphi(x_t', t) \geq -c_3 - \tilde{c}t.} \quad (1)$$

③

For some uniform C_2. ~~Thus~~ Now, let $x_t \in M$ be s.t. $\varphi(x_t, t) = \max_M \varphi(t)$. Then, since $-\Delta_0 \varphi(t) = -\mathrm{tr}_{g(0)} g(t) + n \leq n$.

we have by Green's Formula:

$$\varphi(x_t, t) = \frac{1}{\mathrm{Vol}_{g(0)}(M)} \int_M \varphi(y,t)\, dV_0 - \frac{1}{\mathrm{Vol}_{g(0)}(M)} \int_M \Delta_0 \varphi(y,t)\, G_0(x_t, y)\, dV_0$$

$$\leq \frac{\mathrm{Vol}_0(M \setminus U)}{\mathrm{Vol}_0(M)} \sup_M \varphi(\cdot, t) + \frac{1}{\mathrm{Vol}_0(M)} \int_U \varphi(t)\, dV_0 + C$$

where U is the set where $u \ll 0$ from before. On U we showed $\varphi(t)(z) \leq -\tilde{c} e^t$. Thus

$$\leq \frac{\mathrm{Vol}_0(M \setminus U)}{\mathrm{Vol}_0(M)} \sup_M \varphi - C_4 e^t + C$$

So, as $\mathrm{Vol}_0(M \setminus U) < 1$, we get $\boxed{\max_M \varphi(t) < -C_5 e^t + C_6}$ (2)

This contradicts (1) for t large. Thus u is uniformly bounded below. ∎

Step 2: Bound $|\nabla u|$, R by u.

Proposition: There is a uniform constant C s.t.

$$|\nabla u|^2 \leq C(u+C)$$
$$R \leq C(u+C).$$

pf Let $\Box = \frac{d}{dt} - \Delta$. Then we have the following evolution equations

$$\Box(\Delta u) = -|\nabla\bar{\nabla} u|^2 + \Delta u$$
$$\Box(|\nabla u|^2) = \cancel{\partial} - |\nabla\nabla u|^2 - |\nabla\bar{\nabla} u|^2 + |\nabla u|^2$$

Now set $H = \frac{|\nabla u|^2}{(u+2B)}$ where B is s.t. $u > -B$ $\forall t$ by Step 1. Compute $\Box H$ and apply the maximum principle.

For the second inequality use $G = \frac{-\Delta u}{u+2B}$ and compute $\Box G$, use max. Principle.

Lemma: $\exists$ C uniform such that

$$u(y,t) \leq C\,\mathrm{dist}^2_{g(t)}(x,y) + C$$
$$R(y,t) \leq C\,\mathrm{dist}^2_{g(t)}(x,y) + C$$
$$|\nabla u| \leq C\,\mathrm{dist}^2_{g(t)}(x,y) + C$$

where $u(x,t) = \min_{y \in M} u(y,t)$.

by previous Lemma

pf $\quad |\nabla \sqrt{u+2B}| = \dfrac{|\nabla u|}{2\sqrt{u+2B}} \leq C$, and also

and thus $\quad |\sqrt{u(y,t)} - \sqrt{u(z,t)}| \leqslant \dfrac{|\nabla u|}{2\sqrt{u}}(p,t)\, \mathrm{dist}_{g(t)}(y,z)$

$$\leqslant \tilde{C}\, \mathrm{dist}_t(y,z)$$

where we have set $u = u+2B$ for simplicity.

Thus

$$u(y,t) \leqslant \left(\tilde{C}\, \mathrm{dist}_t(y,x) + \sqrt{u(x,t)}\right)^2$$

$$\leqslant C_1\, \mathrm{dist}_t^2(y,x) + u(x,t)$$

Note that if $u(x_t,t) \longrightarrow \infty$ as $t \to \infty$, then

$$(4\pi)^n = \int e^{-u}\, dV_t \leqslant e^{-u(x_t,t)} \mathrm{Vol}(M) \longrightarrow 0 \text{ as } t \to \infty.$$

Thus $u(x_t,t) \leq K$ for some K independent of time.

Hence $\quad u(y,t) \leqslant c\, \mathrm{dist}_t^2(y,x) + \tilde{C}$

The other two inequalities then follow from the Results of the previous proposition. $\blacksquare$

We have thus Reduced the problem to a uniform diameter bound.

The Key step is the Non-Collapsing Theorem.

Non-Collapsing Theorem.

Prop: Let $g(s)$ be a solution of the unnormalized Kähler Ricci Flow defined on $[0,T)$. There is a constant $\kappa = \kappa(g(0)) > 0$ so that if $|R(g(s))| \le \frac{1}{r^2}$ in a ball $B_{g(s)}(p,r)$, then $\mathrm{Vol}_{g(s)}(B(p,r)) > \kappa r^{2n}$.

Pf Suppose not. Then $\exists$ points $p_k \in M$ and $t_k \in [0,T)$ $t_k \to T$ and radii r_k so that $|R| < \frac{C}{r_k^2}$ but $r_k^{-2n}\mathrm{Vol}(B_k) \to 0$. as $k \to \infty$. Let $\tau = r_k^2$.

Define $u_k(x) = e^{c_k}\varphi(r_k^{-1}\,\mathrm{dist}(x,p_k))$

where φ is C^∞ $\varphi \equiv 1$ on $[0,\frac{1}{2})$ decreasing on $(\frac{1}{2},1]$ $\varphi \equiv 0$ on $[1,\infty)$. We choose c_k so that $(4\pi)^n = e^{2c_k} r_k^{-2n}\int_M \varphi^2\, dV \le e^{2c_k} r_k^{-2n}\mathrm{Vol}(B_k)$

Since $r_k^{-2n}\mathrm{Vol}(B_k) \to 0$ as $k \to \infty$, we must have $c_k \to \infty$.

Claim: we can choose r_k s.t.

(a) $r_k^{-2n} \, Vol(B_k(p_k, r_k)) \to 0$

(b) $r_k^2 R$ is uniformly bdd above on $B(p_k, r_k)$

(c) $$\frac{Vol(B(p_k, r_k))}{Vol(B(p_k, \frac{r_k}{2}))} \leq 3^{2n}.$$

pf ..

Now we compute $W(u_k, g(t_k), r_k^2)$.

$$= (4\pi)^{-n} r_k^{-2n} e^{2c_k} \int (4|\varphi'|^2 - 2\varphi^2 \ln\varphi)\, dv$$

deal.

$$\underbrace{+ r_k^2 \int_B R u^2 (4\pi)^{-n} dv - 2n - 2c_k.}_{\text{bdd.}}$$

we get $W(u_k, g(t_k), r_k^2) \leq C'' - 2c_k \to -\infty$

as $k \to \infty$. But

$$\cancel{A \geq \mu(g(t_k}\ \mu(g(0), t_k + r_k^2) \leq \mu(g(t_k), r_k^2)$$

$$\leq W(u_k, g(t_k), r_k^2) \to -\infty.$$

<u>But</u> $\mu(g(0), t_k + r_k^2) > -\infty$ by results from Last time, and ~~cont~~ continuous dependence on τ of $\mu(g(0), \tau)$, and the fact that $t_k + r_k^2$ lies in a cpct set in $\mathbb{R}^+$.