Lecture: The Kähler-Ricci Flow and the $\bar{\partial}$ operator on Vectorfields.

(2)

Intro: This paper is, in some sense, a continuation of Lecture 1: In this paper we DO NOT assume the uniform Boundedness of the Rm tensor. The trade off is that we lose the Kähler-Gromov cpctness Theorem, and have to assume instead that the smallest pos. eigenvalue λ_t is not degenerating along the flow. Set up: (X, ω_0) compact, Fano.

Recall: NKRF
$$\begin{cases} \dfrac{\partial g_{\bar{k}j}}{\partial t} = g_{\bar{k}j} - R_{\bar{k}j} = -\partial_j \partial_{\bar{k}} u. \\ g_{\bar{k}j}(0) = (g_0)_{\bar{k}j} \in \pi c_1(X) \end{cases}$$

Mabuchi K-energy: Defined by it's value at a point, and it's variation $\delta M(\varphi) = -\frac{1}{V}\int_X \delta\varphi(R-n)\,\omega_\varphi^n$, $V \equiv \int_X \omega_\varphi^n = \pi^n c_1(X)^n$

Recall: In [PS, Stability + Convergence Lecture 1], it was shown that, if $M > -C > -\infty$ on $c_1(X)$, then the function

$$Y(t) = \int_X |\nabla u|^2 \omega_t^n \quad \text{has} \quad Y(t) \longrightarrow 0 \quad \text{as } t \to \infty.$$

Moreover, we computed that

$$\dot{Y}(t) = (n+1)\,Y(t) - \int_X |\nabla h|^2 R\,\omega^n - \int_X |\nabla\bar\nabla h|^2 \omega^n - \int_X |\nabla\nabla h|^2 \omega^n$$

In particular $\displaystyle\int_0^\infty\int_X |D^2 h|^2 \omega^n\,dt = Y(0) + (n+1)\int_0^\infty Y(t)\,dt - \int_0^\infty\int_X |\nabla h|^2 R\,\omega^n$

Now $\mathcal{M}ab > -C \Rightarrow \left|\int_0^\infty Y(t)\right| < \infty$, and Perelman's bound for R $\Rightarrow$ the RHS is finite, and so $\displaystyle\int_0^\infty\int_X |D^2 h|^2 \omega^n\,dt < \infty$

Thus $\|R - n\|_{L^2(\omega(t))} \longrightarrow 0$ as $t \to \infty$. In this paper, the Result is improved!

Theorem 1: Assume that the Mabuchi K-energy is bounded From below on $\pi c_1(X)$. Let $g_{\bar k j}(t)$ be any solution of the Kähler-Ricci flow, and Let $R(t)$ be the Scalar curvature of $g_{\bar k j}(t)$. Then

(i) $\|R(t) - n\|_{C_0} \longrightarrow 0$ as $t \to \infty$

(ii) $\displaystyle\int_0^\infty \|R(t) - n\|^p_{C^0}\,dt < \infty \quad \forall\, p > 2.$

Theorem 2 is an adaptation of the Result in Lecture 1 to the case when $|Rm|$ is NOT uniformly bounded.

Theorem 2: Suppose we have a sol'n of the KRF, $\omega(t)$ the Kähler Forms. Let λ_ω be the smallest positive eigenvalue of $-\bar\partial^\dagger\bar\partial$ acting on $T^{1,0}X$.

(i) If (A) $\displaystyle\inf_{\omega\in\pi c_1(X)} \mathcal{M}(\omega) > -\infty$

Then the $g_{\bar{k}j}$ converge exponentially fast in C^∞ to a KE.

(ii) If $g_{\bar{k}j}(t) \xrightarrow{C^\infty}$ to a KE metric, then (A) and (S) are satisfied.

(iii) In particular, all convergence is exponential.

Before beginning the proof, we recall Perelman's Results.

(if u is normalized by $\frac{1}{V}\int_X e^{-u}\omega^n = 1$, then:

(i) $\exists\, C_0$ depending only on $g_{\bar{k}j}(0)$ s.t.

$$\|u\|_{C^0} + \|\nabla u\|_{C^0} + \|R\|_{C^0} \leq C_0$$

(ii) Let $\rho > 0$ be given. $\exists\, c > 0$ depending only on $g_{\bar{k}j}(0)$, and ρ s.t. $\forall\, x \in X$, and $t \geq 0$ and $r \in (0,\rho]$, we have:

$$\int_{B_r(x)} \omega^n > c r^{2n}$$

where $B_r(x)$ is the geodesic ball of radius r centered at x, wrt $g(t)$.

The Smoothing Lemma:

Lemma 1: $\exists\, \delta, K > 0$ depending only on n s.t. $\forall\, \varepsilon \in (0,\delta]$ and any $t_0 \geq 0$, if $\|u(t_0)\|_{C_0} \leq \varepsilon$, then

$$\|\nabla u(t_0+2)\|_{C^0} + \|R(t_0+2) - n\|_{C^0} \leq K\varepsilon.$$

Key Point: if $\|u\|_{C^0} \longrightarrow 0$, then $\|\nabla u\|, \|\Delta u\| \longrightarrow 0$.

Pf: WLOG, assume $t_0 = 0$. We computed the evolution equation

For u: $\partial_j \partial_{\bar{k}} \dot{u} = \frac{\partial}{\partial t}(g_{\bar{k}j} - R_{\bar{k}j}) = \dot{g}_{\bar{k}j} + \partial_j \partial_{\bar{k}} g^{l\bar{p}} \dot{g}_{\bar{p}l}$

$= \partial_{\bar{k}} \partial_j u + \partial_j \partial_{\bar{k}} \Delta u \overset{\text{Max}}{\underset{\text{Princ.}}{\Longrightarrow}} \boxed{\dot{u} = \Delta u + u - b.}$

where $b = b(t)$ is defined by $b = \frac{1}{V}\int_X u e^{-u} \omega^n$. (This is the necessary Normalization For Perelman's results).

In order to make This Not depend on b, we define $c(t)$ by $\dot{c} = c + b$, $c(0) = 0$. Now set $\hat{u}(t) = u(t) - c(t)$. we have $\|\hat{u}(0)\| \lesssim \varepsilon$, and $\frac{\partial \hat{u}}{\partial t} = \dot{u} - \dot{c} = \Delta u + u + b - c - b$

$= \Delta \hat{u} + \hat{u}$. we Compute:

$$(1) \quad \boxed{\frac{\partial \hat{u}^2}{\partial t} = 2\hat{u}\dot{\hat{u}} = 2\hat{u}(\Delta \hat{u} + \hat{u}) = \Delta(\hat{u})^2 - 2|\nabla \hat{u}|^2 + 2(\hat{u})^2}$$

$$(2) \quad \frac{\partial}{\partial t}|\nabla \hat{u}|^2 = \partial_j \dot{\hat{u}}\, g^{j\bar{k}} \partial_{\bar{k}} \hat{u} + \partial_j \hat{u}\, g^{j\bar{k}} \partial_{\bar{k}} \dot{\hat{u}} - g^{j\bar{p}} \dot{g}_{\bar{p}l} g^{l\bar{k}} \partial_j \hat{u} \partial_{\bar{k}} \hat{u}$$

$$= \partial_j \Delta \hat{u}\, g^{j\bar{k}} \partial_{\bar{k}} u + |\nabla \hat{u}|^2 + \partial_j \hat{u}\, g^{j\bar{k}} \partial_{\bar{k}} \Delta \hat{u} + |\nabla \hat{u}|^2$$

$$F_1$$

$$- g^{j\bar{p}} \partial_{\bar{p}} \partial_l \hat{u}\, g^{l\bar{k}} \partial_j \hat{u} \partial_{\bar{k}} \hat{u}$$

$$\Delta |\nabla \hat{u}|^2 \overset{\text{Norm.}}{\underset{\text{coords.}}{=}} g^{l\bar{p}} \partial_l \partial_{\bar{p}} (\partial_j \hat{u}\, g^{j\bar{k}} \partial_{\bar{k}} \hat{u}) = g^{l\bar{p}} \partial_l [\partial_{\bar{p}} \partial_j \hat{u}\, g^{j\bar{k}} \partial_{\bar{k}} \hat{u}$$

$$+ \partial_j \hat{u}\, \partial_{\bar{p}} g^{j\bar{k}} \partial_{\bar{k}} \hat{u} + \partial_j \hat{u}\, g^{j\bar{k}} \partial_{\bar{p}} \partial_{\bar{k}} \hat{u}]$$

F_2

$$= (\Delta \partial_j \hat{u}) g^{j\bar{k}} \partial_{\bar{k}} \hat{u} + |\bar{\nabla}\nabla \hat{u}|^2 + |\nabla\nabla\hat{u}|^2 + \partial_j \hat{u}\, \partial_\ell \partial_{\bar{p}} g^{j\bar{k}} \partial_{\bar{k}} \hat{u} g^{\ell\bar{p}}$$
$$+ \partial_j \hat{u}\, g^{j\bar{k}} \Delta \partial_{\bar{k}} \hat{u}$$

Thus $\left(\frac{\partial}{\partial t} - \Delta\right)|\nabla \hat{u}|^2 = 2|\nabla\hat{u}|^2 - |\bar{\nabla}\nabla\hat{u}|^2 - |\nabla\nabla\hat{u}|^2 - F_1 - F_2.$

$$\partial_\ell \partial_{\bar{p}} g^{j\bar{k}} = -\partial_\ell \left(g^{j\bar{m}} \partial_{\bar{p}} g_{\bar{m}r} g^{r\bar{k}} \right) \overset{\text{N.C.}}{=} -g^{j\bar{m}} g^{r\bar{k}} \partial_{\bar{p}} \partial_\ell g_{\bar{m}r} =$$

$g^{j\bar{m}} g^{r\bar{k}} R_{\bar{p}\ell\bar{m}r}$ Thus $-g^{\ell\bar{p}} \partial_\ell \partial_{\bar{p}} g^{j\bar{k}} = +g^{j\bar{m}} g^{r\bar{k}} R_{\bar{m}r}$

$$= g^{j\bar{m}} g^{r\bar{k}} \left(-g_{\bar{m}r} + R_{\bar{m}r} + g_{\bar{m}r}\right) = -g^{j\bar{m}} g^{r\bar{k}} \partial_r \partial_{\bar{m}} \hat{u} + g^{j\bar{k}}$$

Thus $F_2 = \partial_j \hat{u}\, \partial_{\bar{k}} \hat{u} \left[-g^{j\bar{m}} g^{r\bar{k}} \partial_r \partial_{\bar{m}} \hat{u} \right] + |\nabla u|^2 = -F_1 + |\nabla \hat{u}|^2$

So

$$\boxed{(2) \quad \left(\frac{\partial}{\partial t} - \Delta\right)|\nabla\hat{u}|^2 = |\nabla u|^2 - |\nabla\bar{\nabla}\hat{u}|^2 - |\nabla\nabla\hat{u}|^2}$$

$$(3) \quad \boxed{\frac{\partial}{\partial t} \Delta \hat{u} = \Delta\Delta\hat{u} + \Delta\hat{u} + |\nabla\bar{\nabla}\hat{u}|^2}$$

(i) From (1) $\frac{\partial (\hat{u})^2_{max}}{\partial t} \leq 2(\hat{u})^2_{max} \implies \frac{\partial}{\partial t}\left((\hat{u})^2_{max} e^{-t}\right) \leq 0.$

Thus $(\hat{u})^2_{max} \leq \varepsilon^2 e^2$ on $[0,2] \implies \|\hat{u}(t)\|_{C^0} < \varepsilon e \quad \forall t \in [0,2].$

(ii) $\frac{\partial}{\partial t}\left(e^{-2t}\left((\hat{u})^2 + t|\nabla\hat{u}|^2\right)\right) \leq \Delta\left(e^{-2t}\left((\hat{u})^2 + t|\nabla\hat{u}|^2\right)\right)$

Applying the maximum p...

$$e^{-2t}\left((\hat{u})^2 + t|\nabla\hat{u}|^2\right) \leq \varepsilon^2 \Rightarrow |\nabla\hat{u}|^2_{C^0} \leq \varepsilon^2 e^4 \quad \forall\, t\in[1,2].$$

using Similar Techniques we can Show:

$$|\Delta\hat{u}|(t) < 2ne^5\varepsilon \quad \text{at} \quad t=2.$$

∎

Recall, $b = \int_X u e^{-u}\omega^n$. In order to prove part (i) of Thm 1 it suffices to show that $\Delta u \to 0$. By Lemma 1, it is enough to show $\|u\|_{C^0} \to 0$. To do this, it suffices to show that b, and $\|u-b\|_{C^0} \to 0$ as $t\to\infty$.

<u>Lemma</u>: (Poincaré Inequality for the measure $e^{-u}\omega^n$)

$$\int_X f^2 e^{-u}\omega^n \leq \int_X |\nabla f|^2 e^{-u}\omega^n + \left(\int_X f e^{-u}\omega^n\right)^2$$

$\forall f\in C^\infty(X)$.

<u>pf</u> Consider the elliptic operator $L := -g^{j\bar{k}}\nabla_j\nabla_{\bar{k}} + g^{j\bar{k}}\nabla_j u\nabla_{\bar{k}}$.

Observe that L is elliptic, and self adjoint wrt the Inner product $\langle\psi,\varphi\rangle = \int \psi\bar{\varphi}e^{-u}\omega^n$. Thus L has all Real eigenvalues. Note that if f is an eigenfunction with λ it's eigenvalue, then

$$\int|\nabla f|^2 e^{-u}\omega^n = \langle Lf, f\rangle = \lambda\int|f|^2 e^{-u}\omega^n.$$

<u>So</u> L has all real, non-negative eigenvalues, and it's

Show that if $f \perp \ker L$ (ie $\int f e^{-u}\omega^n = 0$), then $\lambda \geq 1$. To do this we use Bernsteins trick: Differentiating the equation gives (wrt. $\nabla_{\bar l}$).

$$-g^{j\bar k}\nabla_{\bar l}\nabla_j\nabla_{\bar k} f + g^{j\bar k}\nabla_{\bar l}\nabla_j u\nabla_{\bar k} f + g^{j\bar k}\nabla_j u\nabla_{\bar l}\nabla_{\bar k} f.$$

~~Now, multiply by $g^{m\bar l}\nabla_m f$. we get:~~

$$\cancel{-g^{j\bar k}\nabla_{\bar l}\nabla_j\nabla_{\bar k} f\, g^{m\bar l}\nabla_m f + g^{j\bar k}\nabla_{\bar l}\nabla_j u\nabla_{\bar k} f\nabla_m f g^{m\bar l} + g^{j\bar k}g^{m\bar l}\nabla_j u\nabla_{\bar l}\nabla_{\bar k} f\nabla_m}$$

$$= -g^{j\bar k}\nabla_j\nabla_{\bar i}\nabla_{\bar k} f \qquad + R_{\bar i j}{}^{j}{}_{s}(g^{s\bar k}\nabla_{\bar k} f) + \dots$$

$$= -g^{j\bar k}\nabla_j\nabla_{\bar l}\nabla_{\bar k} f + R_{\bar l}{}^{\bar k}\nabla_{\bar k} f + g^{j\bar k}\nabla_{\bar l}\nabla_j u\nabla_{\bar k} f + g^{j\bar k}\nabla_j u\nabla_{\bar l}\nabla_{\bar k} f$$

Now multiply by $g^{m\bar l}\nabla_m f$ and integrate:

$$\int_X \Big[-g^{j\bar k}\nabla_j\nabla_{\bar l}\nabla_{\bar k} f\, g^{m\bar l}\nabla_m f + \overset{(2)}{g^{m\bar l}R_{\bar l}{}^{\bar k}\nabla_{\bar k} f\nabla_m f} + \overset{(3)}{g^{j\bar k}\nabla_{\bar l}\nabla_j u\nabla_{\bar k} f g^{m\bar l}\nabla_m f}$$

$$+ \int \overset{(4)}{g^{j\bar k}\nabla_j u\nabla_{\bar l}\nabla_{\bar k} f\, g^{m\bar l}\nabla_m f}\Big] e^{-u}\omega^n = \lambda\int_X g^{m\bar l}\nabla_m f\nabla_{\bar l} f e^{-u}\omega^n$$

Integrate by parts in the 1st term. When ∇_j lands on e^{-u}, the result is killed by (4). Also $\nabla_{\bar i}\nabla_j u = g_{\bar i j} - R_{\bar i j}$ so

$$\int_X |\bar\nabla\bar\nabla f|^2 e^{-u}\omega^n + \int_X |\nabla f|^2 e^{-u}\omega^n = \lambda \int_X |\nabla f|^2 e^{-u}\omega^n$$

Thus $\lambda \geq 1$. Now in general, given $f \in C^\infty(M)$, write

$\tilde f = f - \frac{1}{V}\int_X f e^{-u}\omega^n$. Then $\tilde f \perp \ker L$ wrt $\langle\, ,\rangle_u$. $\frac{e^{-u}\omega^n}{V}$

Thus $\tilde f$ is in the span of the eigenfunctions, and we have:

$$\frac{1}{V}\int_X |\nabla \tilde f|^2 e^{-u}\omega^n \geq \frac{1}{V}\int_X |\tilde f|^2 e^{-u}\omega^n = \frac{1}{V}\int |f| e^{-u}\omega^n - \left(\frac{1}{V}\int f e^{-u}\omega^n\right)^2$$

Since $\nabla \tilde f = \nabla f$, we're done $\boxtimes$.

<u>Lemma 3</u>: we have:

(i) $0 \leq -b \leq \|u-b\|_{C^0}$

(ii) $\|u-b\|_{C^0}^{n+1} \leq C \|\nabla u\|_{L^2} \|\nabla u\|_{C^0}^n$

<u>pf</u> Note that the measure $\frac{e^{-u}\omega^n}{V}$ has unit mass, and so we can apply Jensen's Formula. In particular

$$b = \frac{1}{V}\int u e^{-u}\omega^n \leq \log\left(\frac{1}{V}\int e^u e^{-u}\omega^n\right) = 0.$$

<u>Moreover</u>, $\int e^{-u}\frac{\omega^n}{V} = 1$ by assumption. In particular, $\sup_X u > 0$

<u>Thus</u>

$$0 \leq -b \leq \sup_X (u-b) \quad \text{proving (i).}$$

To prove (ii) set $A = \sup_X |u-b| = |u-b|(x_0)$. Since u is C^∞ $\exists$ $r>0$ s.t. $|u-b| > \frac{A}{2}$ on $B_r(x_0)$. Let ρ be as in Perelman's non-collapsing result. Then, clearly we can take $r = \frac{A}{2\|\nabla u\|_{C^0}}$, If $r<\rho$, then

$$\int_{B_r(x_0)} (u-b)^2 \geq \left(\frac{A}{2}\right)^2 c r^{2n} = \frac{A^2}{4} c \frac{A^n}{2^n \|\nabla u\|^{2n}_{C^0}} \geq \tilde{C} \frac{A^{2n+2}}{|\nabla u|^{2n}_{C^0}}$$

<u>Note</u>: Here we have used Perelman's non-collapsing result so that $\tilde{C}$ is independent of time.

<u>Thus</u>

$$\|u-b\|^{n+1}_{C^0} \leq C_1 \|u-b\|_{L^2} \|\nabla u\|^n_{C^0} \underset{\text{(See below)}}{\leq} C \|\nabla u\|_{L^2} \|\nabla u\|^n_{C^0}$$

Now apply Lemma 2: we have:

$$\int_X (u-b)^2 \frac{\omega^n}{V} \underset{(P)}{\overset{\text{Perelman}}{\leq}} C_2 \int (u-b)^2 e^{-u} \frac{\omega^n}{V} \leq C_3 \int |\nabla u|^2 e^{-u} \frac{\omega^n}{V}$$
$$\leq C_4 \int |\nabla u|^2 \frac{\omega^n}{V}.$$

If $r \geq \rho$, then integrate over B_ρ to get an even stronger result.

<u>Rmk</u>: (i) Perelman's Non collapsing estimate is crucial

(ii) we probably didn't need the full strength of Lemma 2.

(iii) If $Mab > -C$, then we know $\|\nabla u\|_{L^2} \to 0$ as the pf of part(i) of theorem (i).

Proof of Theorem 1 part (ii) and Theorem 2.

we begin by proving that stability (A) and (S) yield exponential convergence, which is Part (i) of Thm 2.

we do this by proving 2 Lemmas:

Lemma 5: if $M_{ab} > -C > -\infty$, and $\lambda_t \geq \lambda > 0$, then $\exists\, \mu, C > 0$, independent of t, such that

(i) $Y(t) = \|\nabla u\|_{L^2(t)} \leq Ce^{-\mu t}$

(ii) $\|u\|_{C^0} + \|\nabla u\|_{C^0} + \|R-n\|_{C^0} \leq Ce^{\frac{-1}{2(n+1)}\mu t}$

(i) is obviously necessary. It turns out that (ii) is sufficient.

Lemma 6: Assume $R(t)$ has $\int_0^\infty |R-n|_{C^0}\, dt < \infty$. Then the KRF converges exponentially to a KE metric.

we first prove Lemma 6.

The idea is that the integrability of $|R-n|_{C^0}$ implies a uniform bound for φ (the Kähler Potential).

Let $g_{\bar{k}j} = (g_0)_{\bar{k}j} + \partial_j \partial_{\bar{k}} \varphi$ where we normalize φ by

$$\dot{\varphi} = \log\left[\frac{\det(g_0 + \partial\bar{\partial}\varphi)}{\det(g_0)}\right] + \varphi + u(0) \qquad \varphi(0) = c_0$$

where $c_0 = \int_0^\infty e^{-t} \|\nabla\dot{\varphi}\|^2_{L^2} \, dt + \frac{1}{V}\int_X u(0)\,\omega_0^n$

(why this is a good choice is taken up in Phong, Sturm, Sesum which will be covered Later (?)).

Then we have $\varphi = \dot{\varphi} - \log\left[\frac{\det(g_0+\partial\bar{\partial}\varphi)}{\det(g_0)}\right] - u(0)$

Perelman's Bound for u implies that $\dot{\varphi}$ is bounded. Thus, it suffices to bound the 2^{nd} Term. But we have:

$$\left|\left[\log\frac{\omega^n}{\omega_0^n}\right]\right| = \left|\int_0^t \frac{d}{dt}\left[\log\frac{\omega^n}{\omega_0^n}\right]dt\right| = \left|\int_0^t g^{j\bar{k}}_{(t)}\,\dot{g}_{\bar{k}j}^{(t)}\,dt\right| = \left|\int_0^t R-n\;dt\right| \quad (*)$$

$\leq C < \infty$ where C is independent of time.

Thus $|\varphi|_{C^0} < \tilde{C}$ independent of time. we now have:

$$|\varphi|_{C^0} + |\dot{\varphi}|_{C^0} + |\nabla\dot{\varphi}| + |\Delta\dot{\varphi}| < C \text{ independent of time.}$$

Lemma ([PSS], 2.4)

with our choice of c_0, we have:

$$\sup_{t\geq 0} \|\varphi\|_{C^0} \leq A_0 < \infty \iff \sup_{t\geq 0} \|\varphi\|_{C^k} \leq A_k < \infty$$

$\forall k \in \mathbb{N}$.

= This is essentially the parabolic Analogue of Yau-Aubin estimates.

Once we have uniform bddness of the $\|\varphi\|_{C^k}$, we know that the metrics $g_{\bar{k}j}(t)$ are uniformly bdd. By Arzela-Ascoli we can find times $t_m \to +\infty$ s.t. $\varphi(t_m)$ converge in C^∞ and $g_{\bar{k}j}(t_m)$ converge in C^∞. Observe that Integral bound ~~since the $g_{\bar{k}j}(t_m)$ converge~~ implies, the metrics $g_{\bar{k}j}(t_m)$ are uniformly equivalent.

(ie $\exists C$ s.t. $\frac{1}{C} g_{\bar{k}j}(t_m) \leq (g_0)_{\bar{k}j} \leq C g_{\bar{k}j}(t_m)$)

Now, uniform equivalence + convergence in C^∞ for $g_{\bar{k}j}(t_m)$

$\Rightarrow$ we have uniform curvature bounds for all derivatives

Now, the assumption $\Rightarrow |\Delta_t u|_{C^0} = |R-n|_{C^0} \longrightarrow 0$ as $t_m \to \infty$. But uniform equivalence $\Rightarrow$ this is also true for $|\Delta_0 u|_{C^0}$. By the maximum principle $u(t) \xrightarrow{C^\infty}$ const.

But $\int_X e^{-u}\omega^n = 1$ so $u \xrightarrow{C^\infty} 0$ as $t \to \infty$.

Thus $R^\infty_{\bar{k}j} - g^\infty_{\bar{k}j} = 0$ so $\varphi(\infty)$ is a potential for

Claim: The eigenvalues $\lambda_t \geq \lambda > 0$. as $t \to \infty$.

pf suppose not. Then $\exists$ t_ℓ s.t. $\lambda_{t_\ell} \to 0$. Extract a further subsequence (not relabelled) s.t. $g(t_\ell) \xrightarrow{C^\infty}$ to KE metric. Note that the geometry is uniformly controlled along this sequence:

- Curvature control comes from $g(t_\ell)$ converging in C^∞ and uniform equivalence.
- inj. radius control follows from curvature control and Perelman's uniform diameter bound.

So: we can apply Kähler-Gromov compactness in the special case when J is fixed. (In particular, Notice that in this case, the diffeomorphism group acting on J is trivial, so we have stability (B) Trivially along the subsequence, and so we can apply the results of Lecture 1).

Thus $0 = \lim_{\ell \to \infty} \lambda_{t_\ell} = \lambda(g_\infty) > 0$.

Now, since we know a KE metric exists, we know that $\mu_{ab} > -C > \infty$, and so Lemma 5 $\Rightarrow$

$\|\nabla u\|(t) \to 0$ exponentially.

We now know that the eigenvalue λ_t is NOT degenerating and that $|Rm|$ is uniformly bdd along the flow. We can thus apply the same arguments as in Lecture 1 to show that $\|\nabla u\|_{H^{(s)}(\omega_0)} \xrightarrow{} 0$ and hence b/c we have uniform equivalence.

$|\nabla u|_{C^k} \xrightarrow{\text{exp.}} 0$ for any k by Sobolev imbedding.

But $|\partial_j \partial_{\bar{k}} u|_{C^k} = |\dot{g}_{\bar{k}j}|_{C^k} = |R_{\bar{k}j} - g_{\bar{k}j}|_{C^k} \longrightarrow 0$ exponentially, and so the whole sequence converges. ∎

We now prove Lemma 5:

recall from Lecture 1, we computed that

$$\dot{Y}(t) \leq -2\lambda_t Y - 2\lambda_t \underset{①}{Fut(\pi_t(\nabla^j u))} - \underset{②}{\int_X |\nabla u|^2 (R-n)\omega^n}$$
$$- \underset{③}{\int_X \nabla^j u \nabla^{\bar{k}} u (R_{\bar{k}j} - g_{\bar{k}j})\omega^n}.$$

For our case ① $= 0$ since Mab $> -C > \infty$, ② we can control by Part (i) of Thm 1 (for t large ② $\leq \frac{\lambda}{2}\int |\nabla u|^2 \omega^n$)

However ③, we have No control. Previously (Lec. 1) we were able to control ③ b/c we assumed uniform curvature bounds. In the absence of this assumption we need to work harder.

claim: $\exists\, K_0 > 0$ s.t.

$$\dot{y}(t) \leq -\lambda y(t) + \frac{\lambda}{2} y^{\frac{1}{2}}(t) \cdot \prod_{j=1}^{N} [y(t-a_j)]^{\frac{\delta_j}{2}} \quad \forall t \geq K_0$$

where $N, a_j \in \mathbb{N}$, $\delta_j \in \mathbb{R}^+$ $\sum_{j=1}^{N} \delta_j = 1$.

pf:

we want to estimate $\left| \int \nabla^i u \nabla^{\bar{k}} u \,(R_{\bar{k}j} - g_{\bar{k}j}) \right|$

$$\leq |\nabla u|_{C^0} \left(\int_X |\nabla u|^2 \right)^{\frac{1}{2}} \left(\int_X |R_{\bar{k}j} - g_{\bar{k}j}|^2 \right)^{\frac{1}{2}}$$

now

$$\int_X |R_{\bar{k}j} - g_{\bar{k}j}|^2 = \int_X |\partial_j \partial_{\bar{k}} u|^2 \omega^n \overset{IBP}{=} \int |\Delta u|^2 \omega^n = \int |R-n|^2 \omega^n.$$

$$\left(\int |R-n|^2 \omega^n \right)^{\frac{1}{2}}(t) \leq \sup_X |R-n|(t) \leq K \sup_X |u|(t-2)$$

$$\leq K \sup_X |u-b|(t-2) + |b|(t-2) \overset{\text{Lemma 3}}{\leq} 2K \sup_X |u-b|(t-2)$$

$$\overset{\text{Lemma 3}}{\leq} C \, \|\nabla u\|_{C^0}^{\frac{n}{n+1}}(t-2) \, \|\nabla u\|_{L^2}^{\frac{1}{n+1}}(t-2)$$

So, we have shown that:

$$\left| \int \nabla^i u \nabla^{\bar{k}} u \,(R_{\bar{k}j} - g_{\bar{k}j}) \right| \leq C \, \|\nabla u\|_{C^0}^{\frac{n}{n+1}}(t-2) \, \|\nabla u\|_{C^0}(t) \; y^{\frac{1}{2}}(t) \; y^{\frac{1}{n+1}}(t-2)$$

Note moreover, the last 3 inequalities are general and show that $\|\nabla u\|_{C^0}(t) \leq C_1 \|u-b\|_{C^0}(t-2) \ldots$

$$\leq C_2 \|\nabla u\|_{C^0}^{\frac{n}{n+1}}(t-2)\, \|\nabla u\|_{L^2}^{\frac{1}{n+1}}(t-2) \qquad (\ast)$$

By iterating this inequality we can transfer weight from the C^0 norm to the L^2 norm.

i.e. Let $G(t) = \prod_j A(t-a_j)^{\sigma_j} \prod_k B(t-b_k)^{\Sigma_k}$ where $A(t) = \|\nabla u\|_{L^2}(t)$, $B(t) = \|\nabla u\|_{C^0}(t)$. Note that, initially $\sum \sigma_j = \sigma$, $\sum \Sigma_k = \Sigma$ have $\sigma + \Sigma = 2$ $\left(1 + \frac{n}{n+1} + \frac{1}{n+1} = 2\right)$ $a_j, b_k \geq 0$. Then, iterating $(\ast)$ shows

$G(t) \leq C \tilde{G}(t)$ where $\tilde{G}(t) = \prod_j A(t-\tilde{a}_j)^{\tilde{\sigma}_j} \prod_k B(t-\tilde{a}_k)^{\tilde{\Sigma}_k}$

and $\tilde{\sigma} = \sum_j \tilde{\sigma}_j$, $\tilde{\Sigma} = \sum_k \tilde{\Sigma}_k$ have $\tilde{\sigma} + \tilde{\Sigma} = 2$.

Ex:

$$\|\nabla u\|_{C_0}(t)\, \|\nabla u\|_{C^0}^{\frac{n}{n+1}}(t-2)\, \|\nabla u\|_{L^2}^{\frac{1}{n+1}}(t-2) \leq \|\nabla u\|_{C^0}^{\frac{n}{n+1}}(t-2)\, \|\nabla u\|_{L^2}^{\frac{1}{n+1}}\, \|\nabla u\|_{C^0}^{\frac{n^2}{(n+1)^2}}(t-4)$$
$$\|\nabla u\|_{L^2}^{\frac{n}{(n+1)^2}}(t-4)\, \|\nabla u\|_{L^2}^{\frac{1}{n+1}}(t-2)$$

<u>Note</u>: that $\tilde{\sigma} + \tilde{\Sigma} = 1$ is clear by <u>scaling</u>:

Note that every iteration increases $\tilde{\sigma}$, and so eventually we will have $\tilde{\Sigma} < 1$. Then, set $\delta_j = \frac{\tilde{\sigma}_j}{\tilde{\sigma}}$, and we get

$$\|\nabla u\|_{C^0}(t)\, \|\nabla u\|_{C^0}^{\frac{1}{n+1}}(t-2)\, \|\nabla u\|_{L^2}^{\frac{1}{n+1}}(t-2) \leq H(t) \prod_j A(t-\tilde{a}_j)^{\delta_j}$$

where $H(t) = C \prod_k B(t-\tilde{b}_k)^{\tilde{\varepsilon}_k} \prod_j A(t-\tilde{a}_j)^{\tilde{\sigma}_j - \delta_j}$

<u>Now</u> by construction $\tilde{\sigma}_j - \delta_j > 0$. Thus, by Perelman's bound B is bounded along the flow, and by Lemma 4, we see $H(t) \to 0$ as $t \to \infty$. Thus, we find find K_0 large s.t.

$$\dot{y}(t) \leq -\lambda y(t) + \frac{\lambda}{2} y^{\frac{1}{2}}(t) \prod_{j=1}^{N} [y(t-a_j)]^{\frac{\delta_j}{2}} \quad \forall\ t \geq K_0$$

Let $F(t) = Re^{-\mu t}$ for some $R, \mu > 0$ TBD.

<u>Claim</u>: $y(t) \leq F(t)$

<u>pf</u> suppose not, choose R large s.t. $y(t) < F(t)$ for $t < t_0$ and $y(t_0) = F(t_0)$. Then we can assume that $\dot{y}(t_0) \geq \dot{F}(t_0) = -\mu F(t_0)$. By our previous inequality

$$-\mu F(t_0) \leq -\lambda y(t_0) + \frac{\lambda}{2} y(t_0)^{\frac{1}{2}} \prod y(t_0 - a_j)^{\delta_j} \leq -\lambda F(t_0) + \frac{\lambda}{2} F(t_0)^{\frac{1}{2}} \prod_{j=1}^{N} F(t_0 - a_j)^{\frac{\delta_j}{2}}$$

$$-\mu F(t_0) \leq -\lambda F(t_0) + \frac{\lambda}{2} F(t_0)^{\frac{1}{2}} \prod_{j=1}^{N} F(t_0 - a_j)^{\frac{\delta_j}{2}}$$

$$= -\lambda R e^{-\mu t_0} + \frac{\lambda}{2} R e^{-\frac{\mu t_0}{2}} e^{-\mu \sum_j (t_0 - a_j)\frac{\delta_j}{2}}$$

$$\Rightarrow \quad -\lambda R e^{-\mu t_0} + \frac{\lambda}{2} R e^{-\frac{\mu t_0}{2} + \mu \sum_j \frac{a_j \delta_j}{2}} \geq -\mu e^{-\mu t_0} R$$

$$\Rightarrow \quad + \frac{\lambda}{2} e^{\mu \sum_j \frac{a_j \delta_j}{2}} \geq \lambda + \mu.$$

By choosing μ sufficiently small (a_j, δ_j depend only on n) we can ensure this does not hold. Thus, $y(t) \leq F(t)$ and so exponential decay follows.

Lemma 3 $\Rightarrow$ $\|u\|_{C^0}$ decays exponentially, and finally Lemma 1 $\Rightarrow$ $\|R-n\|_{C^0}$, $\|\nabla u\|_{C^0}$ decay exponentially. ∎

The proof of Theorem 1 part (ii) follows in much the same way. Theorem 2 parts (ii) and (iii) follow in a similar fashion as Lemma 6.