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Lecture: Stability and Convergence of the KRF (Phong, Sturm).

Set up:

Fano (i.e. $c_1(X)>0$, e.g. ample Anti-canonical bundle)

Let (X,ω_0) be a compact Kähler manifold, $\omega_0 \in c_1(X)$.

Introduce two forms of stability:

(A) The Mabuchi energy $\mathcal{V}_{\omega_0}(\varphi)$ is bounded below

(B) Let J be the complex structure of X, viewed as a tensor. Then the C^∞ closure of the orbit of J under the diffeomorphism group of X does not contain any cplx structure J_∞ w/ the property that the space of Hol'c vector fields wrt J_∞ has dimension strictly higher than the dimension of the space of hol'c vector fields wrt. J.

Def'n: The Mabuchi functional $\mathcal{V}_{\omega_0}(\varphi)$ is defined as follows:

Let $\mathcal{H}=\{\varphi \in C^\infty(M,\mathbb{R}) : \omega_\varphi = \omega_0 + i\partial\bar\partial\varphi > 0\}$.

For any $\varphi\in\mathcal{H}$ we define: $\varphi_t : [0,1]\to\mathcal{H}$: $\varphi(0)=0$, $\varphi(1)=\varphi$

$$\mathcal{V}_{\omega_0}(\varphi) = -\int_0^1 \left[\int_X \dot\varphi_t\,(S(\varphi_t)-\bar S)\,\omega_{\varphi_t}^n\right]dt.$$

Note: Mabuchi Functional is independent of the path φ_t.

Moreover, $\mathcal{V}_{\omega_0}(\varphi)$ is defined by it's value at 1-pt. and

$$\frac{\partial \mathcal{V}_{\omega_0}(\varphi)}{\partial t} = -\int \dot\varphi\,(R-n)\,\omega^n.$$

Remark: Condition (B) Fails, then the moduli space of hol'c structures is not Hausdorff. (This is some Form of stability).

We want to prove:

Theorem:

Let (X, J) be cpct, cmplx manifold of dimension n. $\dot{g}_{\bar{k}j} = -R_{\bar{k}j} + g_{\bar{k}j}$ be the NKRF, with initial metric $g_{\bar{k}j}(0)$ in $c_1(X)$. Assume that the Riemann Curvature tensor is uniformly bdd along the flow:

(i) if (A) holds, then for any $s \geq 0$

$$\lim_{t \to \infty} \| R_{\bar{k}j}(t) - g_{\bar{k}j}(t) \|_{(s)} = 0$$

where $\| \cdot \|_{(s)}$ is the Sobolev Norm of order s wrt the metric $g_{\bar{k}j}(t)$.

(ii) if (A) and (B) hold, and if the diameter of X is uniformly bdd above along the flow, then the KRF converges exponentially fast in C^∞ to a KE metric.

Rmk: By Perelman, the diameter and Scalar curvature are uniformly bdd along the KRF.

Proof: Part (i). Some Preliminaries.

Let $h(t)$ be a smooth, real valued function so that

$$\dot{g}_{\bar{k}j} = g_{\bar{k}j} - R_{\bar{k}j} = -\partial_j \partial_{\bar{k}} h \quad (\text{recall } g \in C_1(X) + \text{NKRF preserves } C_1(X))$$

Normalized by

Observe that

$$\partial_j \partial_{\bar{k}} \dot{h} = \frac{\partial}{\partial t}(R_{\bar{k}j}) - \dot{g}_{\bar{k}j} = \left(\partial_j \partial_{\bar{k}} (g^{\ell \bar{p}} \dot{g}_{\bar{p}\ell}) + \partial_j \partial_{\bar{k}} h \right)$$

$$\Rightarrow \quad \partial_j \partial_{\bar{k}} \dot{h} = \partial_j \partial_{\bar{k}} (\Delta h) + \partial_j \partial_{\bar{k}} h$$

so $\quad \dot{h} = \Delta h + h + c \quad$ For some $c = c(t)$.

Evolution Equation for h:

$$\left(\frac{\partial}{\partial t} - \Delta \right) h = + h + c.$$

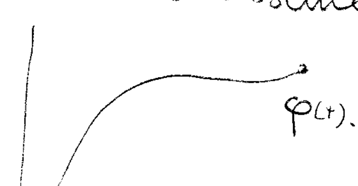
• The Mabuchi Functional along the KRF.

Since $g(t) \in C_1(X)$, we can write $g_{\bar{k}j}(t) = g_{\bar{k}j}(0) + \partial_j \partial_{\bar{k}} \varphi$

Thus, the Evolution $\Rightarrow$ $\partial_j \partial_{\bar{k}} \dot{\varphi} = g_{\bar{k}j}(t) - R_{\bar{k}j} = -\partial_j \partial_{\bar{k}} h.$

Hence $\dot{\varphi} = -h + c'$ where c' depends on t only.

Now, by path independence we can assume the path φ_t is defined by KRF



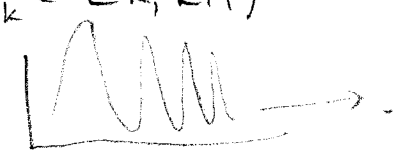
In Particular

$$\dot{\mathcal{V}}_{\omega_0}(\varphi) = -\int_X (-h+c')(\Delta_t h)\,\omega^n_{\varphi(t)} \overset{IBP}{=} -\int_X |\nabla h|^2 \omega^n$$

__Step 1:__ Mabuchi Bounded Below $\Rightarrow$ Sobolev Convergence.

Mabuchi Bounded below $\Rightarrow$ $\displaystyle\int_0^\infty \int_X |\nabla h|^2 \omega^n < \infty.$

__Thus__, $\exists \{t_k\}$ $t_k \to \infty$ such that $y(t) = \int_X |\nabla h|^2 \omega^n$, has $y(t_k) \to \infty$. and $t_k \in [k, k+1)$

Need to Rule out : .

compute the Evolution equation for $y(t)$.

h $\mathbb{R}$ valued.

$$\dot{y}(t) = \frac{d}{dt}\int_X \partial_j h\, g^{j\bar k}\, \overline{\partial_k h}\, \omega^n = \int_X \partial_j \dot h\, g^{j\bar k}\, \partial_{\bar k} h$$

$$- \int_X \partial_j h\, \partial_{\bar k} h\, g^{j\bar l}\, \dot g_{\bar l p}\, g^{p\bar k} + \int_X \partial_j h\, g^{j\bar k}\, \partial_{\bar k} \dot h + \int_X \partial_j h\, g^{j\bar k}\, \partial_{\bar k} h\, \dot u$$

__Exercise__ :

(i) $(\dot{\partial_j h}) = \Delta(\partial_j h) + \partial_j h.$

(ii) $$\Delta |\partial_j h|^2 = g^{j\bar k}\Delta(\partial_j h)\,\partial_{\bar k} h + g^{j\bar k}\partial_j h\, \Delta \partial_{\bar k} h + R^{j\bar k}\partial_j h\, \partial_{\bar k} h + |\bar\nabla \nabla h|^2 + |\nabla\nabla h|^2$$

(iii) $$\dot{|\partial_j h|^2} = g^{j\bar k}(\dot{\partial_j h})\,\partial_{\bar k} h + g^{j\bar k}\partial_j h\, \partial_{\bar k}\dot h + R^{j\bar k}\partial_j h\, \partial_{\bar k} h - |\nabla h|^2$$

Finally, we get:

$$\left(\frac{\partial}{\partial t} - \Delta\right)|\nabla h|^2 = -|\bar{\nabla}\nabla h|^2 - |\nabla\nabla h|^2 + |\nabla h|^2$$

Now: Since $(\dot{\omega^n}) = (-R+n)\omega^n$, we get:

$$\dot{y}(t) = \int |\dot{\nabla h}|^2 \omega^n + \int |\nabla h|^2 (n-R)\omega^n$$

$$\boxed{\dot{y}(t) = \int \Delta|\nabla h|^2 - |\bar{\nabla}\nabla h|^2 - |\nabla\nabla h|^2 + |\nabla h|^2 + \int |\nabla h|^2 (n-R)} \quad (\star)$$

$$= (n+1)y(t) - \int R|\nabla h|^2 - \int |\bar{\nabla}\nabla h|^2\omega^n - \int |\nabla\nabla h|^2\omega^n$$

By Perelman we have $|R| \leq C$. And so:

$$\dot{y}(t) \leq (n+1+C)\,y(t) \Rightarrow y(t) \leq y(s)e^{((n+1)+C)(t-s)} \quad \forall t \geq s.$$

Hence $y(t) \leq y(t_m)e^{(n+1)+C} \quad \forall\ t \in [m, m+1)$. Thus $y(t) \to 0$.

Now, integrating $(\star)$ gives:

$$\int_0^\infty dt \int_X |\bar{\nabla}\nabla h|^2 + \int_0^\infty dt \int_X |\nabla\nabla h|^2 = y(0) + (n+1)\int_0^\infty y(t)\,dt - \int_0^\infty dt \int_X R|\nabla h|^2\omega^n$$

Note that the RHS is bounded since:

(i) $|R| \leq C$ uniformly along the flow

(ii) The Mabuchi Functional is uniformly bdd <u>below</u>.

In particular $|\nabla\nabla h|$, $|\nabla\bar\nabla h| \longrightarrow 0$ along a subsequence.

Now, Repeat the argument: ie compute the evolution eqn's. They involve Lower order terms, and curvature terms. (including Derivatives of the curvature Tensor).

Theorem (Hamilton, Shi)
The uniform bddness of the curvature Tensor along the RF $\Rightarrow$ uniform bddness of all the covariant derivatives of Riemann of any fixed order. [Ref 20, 32 in PS].

Now use an inductive argument.

Part 2 of Theorem (i):

Tempting: uniform bddness of the curvature + uniform bddness of diameter $\Rightarrow$ uniform bddness of injectivity radius [cheege
Since Volume is preserved under NKRF, uniform diameter and injectivity radius control $\Rightarrow$ uniform control of the Sobolev constant. Thus

$$\sup_X |D^p \dot g_{\bar k j}(t)|_t = \sup_X |D^p (R_{\bar k j} - g_{\bar k j}(t))|_t \to 0.$$

For any p. The problem is the norms are wrt the evolving metric.

Compactness Theorems in Geometry and App. to KRF.

Defn of convergence: (C^p-convergence)

Let $K \subseteq M$ be a compact set, and let $\{g_k\}_{k\in\mathbb{N}}$, g_∞, and g be Riemannian metrics on M. For $p \in \{0\} \cup \mathbb{N}$, we say that g_k converges in C^p to g_∞ uniformly on K if, $\forall \varepsilon > 0$ $\exists$ $k_0 = k_0(\varepsilon)$ s.t. $\forall k \geq k_0$

$$\sup_{0 \leq \alpha \leq p} \sup_{x \in K} \left| \nabla^\alpha (g_k - g_\infty) \right|_g < \varepsilon$$

where ∇ is wrt the ref. metric g.

Note: Since we are on a cpct set, the choice of metric g on K does not affect the convergence. (ie we can take $g = g_\infty$)

Defn: (Pointed manifolds and Solutions)

A pointed Riemannian mfd is a 3-tuple (M^n, g, O) where (M,g) is a Rmn mfd, $O \in M$ is a choice of pt. (called the origin, or basepoint). If the metric g is complete, then (M,g,O) is a complete PRM. $(M^n, g(t), O)$ is a pointed Solution to the KRF if $(M, g(t))$ is a soln of the Ricci flow.

$\{(M^n_k, g_k, O_k)\}$ complete PRM converge to a complete PRM $(M^n_\infty, g_\infty, O_\infty)$ if $\exists$

(i) an exhaustion $\{U_k\}_{k\in\mathbb{N}}$ of M_∞ by open sets with $O_\infty \in U_k$

(ii) a sequence of diffeomorphisms $\Phi_k : U_k \to V_k := \Phi_k(U_k) \subseteq M_k$ with $\Phi_k(O_\infty) = O_k$

s.t. $(U_k, \Phi_k^*[g_k|_{V_k}])$ converges in C^∞ to (M_∞, g_∞) uniformly on cpct sets in M_∞.

Rmk: Can make the def'n for complete Pointed Solutions to KRF.

Theorem (Hamilton)(Compactness For metrics)

Let $\{(M^n_k, g_k, O_k)\}_{k\in\mathbb{N}}$ be a sequence of CPRM that satisfy

(1) (uniformly bdd geometry)

$$|\nabla_k^p Rm_k|_k \leq C_p \text{ on } M_k$$

$\forall\, p \geq 0$, and k; where $C_p < \infty$ is a sequence of constants independent of k.

(2) (injectivity radius estimate)

$$inj_{g_k}(O_k) \geq i_0$$

For some $i_0 > 0$. Then $\exists$ a subsequence $\{j_k\}$ s.t.

$\{(M$ ...

Now, our curvature assumption $\Rightarrow$ we have bdd geometry.
Thus, we can apply Hamilton's theorem to find diffeomorphisms F_{t_j} s.t. $(F_{t_j}^* g(t_j))$ converge in C^∞.
However, we have NO control over the diffeomorphisms and so cannot deduce any convergence results.

Recall: we are Assuming:

(A) Mabuchi Functional Bdd below

(B) Condition on orbit of J under diffeos.

Step 1: exponential decay of $|\dot{g}_{\bar{k}j}(t)|_t$. Let $y = \int_X |\nabla h|^2 \omega^n$ as before. We compute:

$$\dot{y}(t) = -\int_X |\nabla h|^2 (R-n)\,\omega^n - \int_X \nabla^j h \nabla^{\bar{k}} h\,(R_{\bar{k}j} - g_{\bar{k}j})\,\omega^n - 2\int_X |\bar{\nabla}\bar{\nabla} h|^2 \omega^n$$

pf:

$$\dot{y}(t) = -\int_X |\nabla h|^2 (R-n)\,\omega^n + \int |\nabla \dot{h}|^2 \omega^n$$

Now apply prev. computation:

$$\int |\nabla \dot{h}|^2 \omega^n = -\int |\bar{\nabla}\nabla h|^2 - \int |\nabla\nabla h|^2 + \int |\nabla h|^2$$

Bochner-Kodaira Formula for $T^{1,0}$

$$\|\nabla V\|^2 = \|\bar{\nabla} V\|^2 + \int_X R_{\bar{k}j} V^j \overline{V^k}\,\omega^n. \quad \text{(exercise: compute this)}$$

Apply to $V^j = \nabla^j h = (\bar{\nabla} h)^j$. Then we get.

$$\int |\bar\nabla\nabla h|^2 = \int |\nabla\bar\nabla h|^2 = \int |\nabla V|^2 = \int |\bar\nabla V|^2 + \int_X R_{\bar k j} V^j \overline{V^k}\,\omega^n$$

$$= \int |\bar\nabla\bar\nabla h|^2 + \int_X R_{\bar k j} V^j \overline{V^k}\,\omega^n. \qquad \left(\underline{\text{Note}}\ |\bar\nabla\bar\nabla h|^2 = |\nabla\nabla h|^2\right)$$

(b/c $h : X \to \mathbb{R}$).

So

$$\boxed{\dot y(t) = -\underset{(1)}{\int |\nabla h|^2 (R-n)} - \underset{(2)}{\int_X \nabla^j h\, \nabla^{\bar k} h\, (R_{\bar k j} - g_{\bar k j})\,\omega^n} - 2\int_X |\bar\nabla\bar\nabla h|^2_{\omega}} \qquad (44)$$

Note: The only hope for getting exponential decay is to get a strictly positive lower bound for $\int_X |\bar\nabla\bar\nabla h|^2 \omega^n = \int |\bar\partial V|^2 \omega^n$

~~b/c In general, we can't control terms (1) and (2).~~

Let λ_t be the lowest positive eigenvalue of $-\bar\partial^{+}\bar\partial$ on $T^{1,0}X$.

By Elliptic Theory, we have:

$$\lambda_t \| V - \pi_t V \|^2 \leq \int_X |\bar\nabla V|^2 \omega^n \qquad (\text{see, e.g. pf of Hodge Theorem}).$$

where π_t is the orthogonal projection wrt. $g_{\bar k j}(t)$ onto the space $H^0(T^{1,0}X)$

Claim: $\| \pi_t V \|^2 = fut(\pi_t V)$ when $V^j = \nabla^j h$.

Defn: The Futaki Invariant of X, is the character defined on $H^0(T^{1,0})$ by $\quad fut(W) = \int_X (Wh)\,\omega^n$

pf

$$\langle \pi_t V, \pi_t V\rangle = \langle V, \pi_t V \rangle = \int g^{j\bar k}\partial_{\bar k} h\, \overline{(\pi_t V)^p g_{\bar p j}}$$

$$= \int \overline{(\pi_t V)^p \partial_p h} = \int (\pi_t V)^p \partial_p h = fut(\pi_t V). \qquad \square$$

$$\dot{y} \leq -2\lambda_t y + 2\lambda_t \, \mathrm{Fut}(\pi_t V) - \int (R-n)|\nabla h|^2 - \int \nabla^j h \nabla^{\bar{k}} h (R_{\bar{k}j} - g_{\bar{k}j})$$

NB: This equation is completely general. We have not used any assumptions!

Theorem: (Tian)

Mabuchi Functional bounded below $\Rightarrow$ Futaki invariant vanishes.

Also
$$\int \nabla^j h \nabla^{\bar{k}} h (R_{\bar{k}j} - g_{\bar{k}j}) \leq |R_{\bar{k}j} - g_{\bar{k}j}|_t \, y(t)$$

Now: Theorem (1) pt. 1 $\Rightarrow$ $\forall \varepsilon > 0$ $\exists T_\varepsilon$ Large s.t. if $t \in [T_\varepsilon, \infty)$

Then $|R_{\bar{k}j} - g_{\bar{k}j}|_t < \frac{\varepsilon}{2}$, and $|R - n| \leq \frac{\varepsilon}{2}$. Then

$$\dot{y}(t) \leq (-2\lambda_t + \varepsilon)\, y(t).$$

Thus: exponential convergence follows if we can bound λ_t below by a positive constant.

Note: λ_t is NOT accessible via Bochner-Kodaira Techniques as these apply to Negative Bundles, and here we have a positive bundle.

the cplx structure J is stable in the sense of (B). Fix $V>0$, $D>0$ $\delta>0$, and constants C_k. Then $\exists$ N, and $C = C(V, D, \delta, C_k, n, N)$ s.t.

$$C\|W\|^2 \leq \|\bar{\partial} W\|^2 \quad , \quad W \perp H^0(T^{1,0}X)$$

$\forall$ Kähler metrics g with $\mathrm{Vol}_g(X) \leq V$, $\mathrm{Diam}_g(X) \leq D$, $\mathrm{inj}_g X \geq \delta$ and $|\nabla^k Rm| \leq C_k$ $\forall k \leq N$.

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with Theorem 3, $\exists$ $c>0$ s.t. $\dot{y}(t) \leq -c\, y(t)$ for t suff Large.
Then $y(t) \leq C e^{-ct}$

A similar inductive argument as before shows that

$$\|\bar{\nabla}^r \nabla^s h\|_{L^2(X, g(t))} \longrightarrow 0 \quad \text{exponentially Fast.}$$

- Exponential decay + Sobolev Imbedding with uniform constants $\Rightarrow$ $\sup_X |\nabla^k h|_t^2 \leq C_k e^{-t}$.
- Now show the metrics are uniformly equivalent ie $\exists$ C s.t. $\frac{1}{C}(g_0)_{\bar{k}j} \leq g_{\bar{k}j}(t) \leq C (g_0)_{\bar{k}j}$ $\quad C$ indep. of t.

That is: They induce the Same Topology in C^k $\forall k$.

Lemma (Hamilton)
It suffices to show $\displaystyle\int_T^\infty \sup_X |\dot{g}_{\bar{k}j}|_t \, dt < \infty$. $\quad (*)$

pf

$$\left| \log \frac{\det(g_S)}{\det(g_T)} \right| = \left| \int_T^S [\log \det(g(t))]' \, dt \right| = \left| \int_T^S g^{j\bar{k}} \dot{g}_{\bar{k}j} \, dt \right| \leq \int_T^S \sup_X |\dot{g}_{\bar{k}j}| \, dt < C$$

pf Fix $p \in X$. $W \in T_p X$. Then $\forall\, S \geq T$

$$\left| g_{\bar{k}j}(T) W^j \overline{W}^k - g_{\bar{k}j}(S) W^j \overline{W}^k \right| = \left| \int_T^S \dot{g}_{\bar{k}j}(t)\, W^j \overline{W}^k \, dt \right|$$

$$\leq \int_T^S \sup_X |\dot{g}_{\bar{k}j}|_t \, |W|_t^2 \, dt \leq C |W|^2_{t=0} \int_T^S e^{-ct} dt$$

$$\leq C' |W|^2_{t=0} \left(e^{-cS} - e^{-cT} \right) \longrightarrow 0 \text{ exponentially}$$

as $S, T \to \infty$. Thus the metrics converge, and since $\partial\bar{\partial} h \to 0$ the limit is KE. ∎

It remains to prove Theorem 3. To do this we need the following Kähler version of Gromov Compactness

Theorem 4: Let X be a cpct smooth mfd. Let $(g(t), J(t))$ be any sequence of metrics & cplx structures on X s.t. $(X, g(t), J(t))$ is Kähler. Assume $g(t)$'s have bounded geometry. Then $\exists$ $\{t_j\}$ and diffeos $F_{t_j}: X \to X$ s.t. $\tilde{g}(t_j) = F^*_{t_j}(g_{t_j})$ converge in C^∞ to a smooth metric $\tilde{g}(\infty)$, and the pull-back complex structures $\tilde{J}(t_j) = F^*_{t_j} J(t_j)$ converge in C^∞ to an integrable cplx structure tensor $\tilde{J}(\infty)$. Furthermore, $\tilde{g}(\infty)$ is Kähler wrt $\tilde{J}(\infty)$.

pf the C^∞ part is just Hamilton's version of Gromov cpctness. Thus, we may assume that $\tilde{g}(t_j)$ converge, and focus on showing the cplx structures converge along a subsequence

To fix notation, let $J = J(\infty)$, write $J_t = J(t)$ and $g_t = g(t)$.
Let ∇ be the connection associated to g (Levi-Civita).
The Idea is to use Arzela-Ascoli to construct a C^∞ limit.
Ie. $\forall \alpha$, we want to find C_α s.t. $|\nabla^\alpha J_t| \leq C_\alpha \ \forall t$.

Note: Since (M, g_t, J_t) is Kähler, we have: $g_t(u,v) = g_t(Ju, Jv)$.
ie $g_{ij} = J_i^k g_{k\ell} J_j^\ell$. $J^2 = -\mathbb{1}$.

(i) $\alpha = 0$. Working in Normal coordinates for g_t gives

$$1 = J_i^k g_{k\ell} J_i^\ell = \sum_k (J_i^k)^2. \quad \text{Thus } |J_t|_{g_t} = n.$$

Now, Since $g_t \to g$, this yields $|J_t|_g < C_0$

(ii) Now prove by induction. $\alpha = 0$ done. Note that (M, g_t, J_t) Kähler $\Rightarrow$ $\nabla_t J_t = 0$. Thus $\nabla^\alpha J_t = \nabla^{\alpha-1}(\nabla - \nabla_t) J_t$ $(***)$

Let $H_t = \nabla - \nabla_t$. That is $(H_t)_{ij}^p = \Gamma_{ij}^p - \Gamma(t)_{ij}^p$

Then $(H_t)_{ij}^p g_{p\ell}^{(t)} + (H_t)_{i\ell}^p g_{jp}^{(t)} = \nabla_i g_{j\ell} - (\nabla_t)_i g_{j\ell} = \nabla_i g_{j\ell}$.
Using the symmetry of H_t in the lower indices, we get

$$2(H_t)_{ij}^p = (g_t)^{pk}\left[\nabla_j (g_t)_{ki} + \nabla_i (g_t)_{jk} - \nabla_k (g_t)_{ij}\right]$$

Hence (H_t) is bdd uniformly in C^∞, and converges in
∞ ...

It Follows from ($\ast\ast\ast$) that J_t converges along a subsequence to J_∞ and J_∞ is clearly a cplx structure. Moreover

$$H_t J_t = \nabla J_t \quad \Rightarrow \quad H_\infty \tilde{J}_\infty = \nabla J_\infty . \text{ But } H_\infty = 0$$

Thus $\tilde{J}_\infty$ is Kähler, and we're done ▨

We now prove Theorem 3.

Proposition: if $(g(t), J(t)) \xrightarrow{C^\infty} (g_\infty, J_\infty)$, and if the dimension of the space of Holomorphic vectorfields is the same $\forall\, t \in [N, \infty]$ then $\lim_{t \to \infty} \lambda_t = \lambda_\infty$. where λ_t is the smallest strictly positive eigen value of $-\Delta_t$.

Assume the Proposition. We Prove theorem 3 by contradiction. If Thm 3 does not hold, then $\exists$ a subsequence of metrics $g(t)$ with $\lambda_t \to 0$. By passing to a further subsequence, we may apply the Result of Theorem 4 to find diffeomorphisms $F_t : X \to X$ s.t. $(F_t^* g_t, F_t^* J_t)$ converge in C^∞ to (g_∞, J_∞). Now By the proposition, if $\tilde{\lambda}_t$ is the smallest pos. eigenvalue of $(\tilde{g}_t, \tilde{J}_t) = (F_t^* g_t, F_t^* J_t)$, then $\tilde{\lambda}_t \to \tilde{\lambda}_\infty$. $\lambda_\infty > 0$. But F_t is a biholomorphic isometry (by construction) and so $\tilde{\lambda}_t = \lambda_t$, and we get a contradiction. ▨

pf of the proposition: We just outline the key steps:

Let $\|\cdot\|_{H_t^{(s)}}$ be the Sobolev norm of order s on $T(X)$

Then, since $g(t) \to g_\infty$, and the metrics are uniformly equivalent, we have:

(i) $\left| \langle u,v\rangle_t - \langle u,v\rangle_\infty \right| \leq c_t \|u\|_\infty \|v\|_\infty \qquad c_t \to 0$

(ii) $C_t^{-1} \|v\|_{H_t^{(s)}} \leq \|v\|_{H_\infty^{(s)}} \leq C_t \|v\|_{H_t^{(s)}} \qquad C_t \to 1$

(iii) $\exists\ C$ uniform, s.t. the Elliptic Apriori Estimate Holds

$$\|v\|_{H_t^{(1)}} \leq C\left\{ \langle \Delta_t v, v\rangle_t + \|v\|_{H_t^{(0)}} \right\}. \qquad \forall\ v \in C^\infty(X, T^{1,0}X).$$

Step 1: Let $\{\varphi_t^\alpha\}_{1\leq\alpha\leq N}$ be a basis of O.N. eigenvectors for Δ_t. ie $\Delta_t \varphi_t^\alpha = 0$, $\langle \varphi_t^{(\alpha)}, \varphi_t^{(\beta)}\rangle_t = \delta^{\alpha\beta}$.

A priori estimate $\Rightarrow$ $\|\varphi_t^\alpha\|_{H_t^{(1)}} \leq C \quad \forall\ \alpha$.

Now (ii) $\Rightarrow$ $\{\varphi_t^\alpha\}$ is uniformly bdd in $H_\infty^{(1)}$. So by Rellich's Lemma $\exists$ $\{\varphi_{t_j}^\alpha\}$ convergent in L^2_∞.

and $\langle \varphi_\infty^{(\alpha)}, \varphi_\infty^{(\beta)}\rangle = \delta^{\alpha\beta}$. Also, $\Delta_\infty \varphi_\infty^{(\alpha)} = \lim_{j\to\infty} \Delta_t \varphi_t^{(\alpha)}$

Thus, $\varphi_\infty^{(\alpha)}$ are weakly Harmonic, then elliptic Regularity

$\Rightarrow$ $\varphi_\infty^{(\alpha)}$ are C^∞ and Harmonic, and they are an O.N. set in $H^0_\infty(X, T^{1,0}X)$.

Step 2: Since $\dim K_t = \dim \operatorname{Ker}(\Delta_t) = \dim \operatorname{Ker} \Delta_\infty = \dim K_\infty$ by assumption, we know that $\{\varphi_t^{(\alpha)}\}_{1 \le \alpha \le N}$ converges to an O.N. basis of K_∞. Let $K_t^\perp$ be the orthogonal comp. of K_t, and $\psi_t \in K_t^\perp$ be a lowest eigenfunction.

$\Delta_t \psi_t = \lambda_t \psi_t$, $\psi_t \in K_t^\perp$, $\|\psi_t\|_{H_t^{(2)}} = 1$.

Assume $\liminf_{t \to \infty} \lambda_t < \lambda_\infty$: $\varepsilon > 0$, $\exists$ subsequence (not relabeled) s.t. $\lambda_t \le (1-\varepsilon)\lambda_\infty$. Then $\|\Delta_t \psi_t\|^2_{L^2} = \lambda_t$ is bounded, and so by elliptic A priori estimate ψ_t is uniformly bdd in H^2_t. By considering Δ_t^2, and it's A priori estimate, we see that ψ_t is uniformly bdd in $H_t^{(4)}$, and hence we can assume ψ_t converges in $H_\infty^{(2)}$. $\psi_t \to \psi_\infty$ in $H_\infty^{(2)}$. Clearly $\psi_\infty \perp K_\infty$ (use step 1). Let π be the orthogonal projection from K_∞ to $K_\infty^\perp$. Then

$$\left.\begin{array}{l} (g(t), J(t)) \to (g_\infty, J_\infty) \\ \{\varphi_t^\alpha\} \to \{\varphi^\alpha\} \end{array}\right\} \Rightarrow \|\pi\psi_t - \psi_t\|_{H^2_\infty} \to 0$$

Now $\| \Delta_t - \Delta_\infty \|_{Hom(H^{(2)}_\infty, H^{(0)}_\infty)} \longrightarrow 0$

So

$$\langle \Delta_t \psi_t, \psi_t \rangle_t = \langle \Delta_\infty (\Pi \psi_t), \Pi \psi_t \rangle_\infty - o(1) \geq \lambda_\infty \| \Pi \psi_t \|^2_\infty - o(1)$$

where $o(1) \to 0$ as $t \to \infty$.

Thus $\lim_{t \to \infty} \| \Pi \psi_t \|^2_\infty < 1$, But $\| \Pi \psi_t \|^2_{L^2_\infty} \longrightarrow 1$ so we have a contradiction.

what's the Point:

if $\exists\, t$ with $\lambda_{t_j} \leq (1-\varepsilon)\lambda_\infty$, then we can construct a convergent subsequence in H^2_t with eigenvalue $< \lambda_\infty$ and $\Delta_t \psi_t \longrightarrow \Delta_\infty \psi_\infty$, which contradicts λ_∞ being the Lowest eigenvalue.

So: How does stability manifest itself?

(1) the Mabuchi Functional bounded below allows us to get H^p_t convergence.

Problem: The spaces H^p_t are NOT uniformly equivalent.

(2) Stability (B) allows us to get exponential decay of all derivatives of the Ricci Potential. In particular, we get uniform equivalence.

Once we have uniform equivalence, we can put our metrics in a single Sobolev space, and use Sobolev imbedding to get C^∞ convergence.

How do we show that (B) $\Rightarrow$ the Lowest eigenvalue is not degenerating?

(i) Assume it is: Kähler-Gromov compactness allows to extract a subsequence. This subsequence converges in the Cheeger-Gromov sense. (B) allows us to prove the Proposition $\lim_{t\to\infty} \lambda_t = \lambda_\infty > 0$. Now the diffeos in Kähler-Gromov are biholomorphic isometries, so the eigenvalues don't change, and we have a contradiction.

(ii) How does (B) allow us to prove the prop? Since $\dim \ker \Delta_t = \dim \ker \Delta_\infty$ we can construct an O.N. basis for $\ker \Delta_t$ that converges to an O.N. basis for $\ker \Delta_\infty$. This allows us to SPLIT $H_t^{(s)}$ along the sequence $t \to \infty$.