Adam Talk

Theorem 1 : No holo VF, $c_1 > 0$, X admits KE $\iff$

$$(*) \quad \frac{1}{V}\int_X e^{-\phi}\omega_{KE} \le C e^{(1-A)J_{KE}(\phi) - \frac{1}{V}\int_X \phi\, \omega_{KE}^2}, \quad \forall \phi \in P(X, \omega_{KE})$$

- $$F_\omega(\phi) = J_\omega(\phi) - \frac{1}{V}\int_X \phi\, \omega^n - \log\left(\frac{1}{V}\int_X e^{h_\omega - \phi}\omega^n\right)$$
- $$J_\omega(\phi) = \frac{i^{n-1}}{2V}\sum_{j=0}^{n-1}\left(\frac{n-j}{n+1}\right)\int_X \partial\phi\wedge\bar\partial\phi\wedge\omega^{n-1-j}\wedge\omega_\phi^j$$
- $$Ric(\omega) - \omega = \frac{i}{2}\partial\bar\partial h_\omega$$
- $$\int_X e^{h_\omega}\omega^n = V$$

$$(*) \quad \boxed{F_{\omega_{KE}}(\phi) \ge A\, J_{\omega_{KE}}(\phi) - B}$$

Step 1 : Prove $F_{\omega_{KE}}(\phi) \ge A_\gamma \left(J_{KE}(\phi)\right)^\gamma - B_\gamma$ $(**)$

assuming $osc(\phi) \le K(J_{KE}(\phi) + 1)$

Step 2 : Prove $(*)$.

Fix ϕ, set $\omega = \omega_{KE} + \frac{i}{2}\partial\bar\partial\phi$.

Exists because of Bando-Mabuchi

(MA) $(\omega + \frac{i}{2}\partial\bar\partial\phi_t)^n = e^{h_\omega - t\phi_t}\omega^n$, $t \in [0,1]$

$t=1 \Rightarrow -Ric(\omega_{\phi_1}) + Ric(\omega) = \frac{i}{2}\partial\bar\partial h_\omega - \frac{i}{2}\partial\bar\partial\phi_1$

$\omega_{KE} = \omega + \frac{i}{2}\partial\bar\partial\phi_1 = Ric(\omega_{\phi_1})$

$\Rightarrow \phi_1 = -\phi + c$

$$J_\omega(\phi) = \frac{i}{2V}\sum_{j=0}^{n-1}\int_X \partial\phi\wedge\bar\partial\phi\wedge\omega^j\wedge\omega_\phi^{n-1-j}$$

$$= \frac{1}{V}\int\phi\underbrace{\left(-\frac{i}{2}\partial\bar\partial\phi\right)}_{\omega-\omega_\phi}\sum_{j=0}^{n-1}\omega^j\wedge\omega_\phi^{n-j-1}$$

$$= \frac{1}{V}\int_X\phi(\omega^n - \omega_\phi^n)$$

$$\frac{d}{dt}I_\omega(\phi_t) = \frac{1}{V}\int_X\dot\phi_t(\omega^n - \omega_{\phi_t}^n) - \frac{1}{V}\int_X\phi_t\frac{d}{dt}(\omega_{\phi_t}^n)$$

$$\frac{d}{dt}J_\omega(\phi_t) = \frac{1}{V}\int_X\dot\phi(\omega^n - \omega_{\phi_t}^n)$$

$$\frac{d}{dt}(I_\omega - J_\omega)(\phi_t) = -\frac{d}{dt}\left(\frac{1}{V}\int_X \phi_t\, \omega_{\phi_t}^n\right) + \frac{1}{V}\int_X \dot{\phi}_t\, \omega_{\phi_t}^n$$

Thing 1 show ≥ 0 along (MA)

Thing 2 get expression for $F_{KE}(\phi)$ nicer

$$V = \int_X \omega_{\phi_t}^n = \int_X e^{h_\omega - t\phi_t}\, \omega^n$$

derivative???

$$0 = \int_X (-t\dot{\phi}_t - \phi_t)\, \omega_{\phi_t}^n$$

$$\frac{d}{dt}\left[I_\omega - J_\omega(\phi_t) + \frac{1}{V}\int_X \phi_t\, \omega_\phi^n\right] = -\frac{1}{tV}\int_X \phi_t\, \omega_{\phi_t}^n$$

$$\Rightarrow t\frac{d}{dt}\left[I_\omega - J_\omega(\phi_t) + \frac{1}{V}\int_X \phi_t\, \omega_{\phi_t}^n\right] + \frac{1}{V}\int_X \phi_t\, \omega_{\phi_t}^n = 0$$

$$(I_\omega - J_\omega)(\phi_t) + \frac{1}{V}\int_X \phi_t \, \omega^n_{\phi_t} + t \frac{d}{dt}\left[(I_\omega - J_\omega)(\phi_t) + \frac{1}{V}\int_X \phi_t\right.$$

$$\Rightarrow (I_\omega - J_\omega)(\phi_t) + \frac{1}{V}\int_X \phi_t \, \omega^n_{\phi_t} = \frac{1}{t}\int_0^t (I_\omega - J_\omega)(\phi_s)\,ds$$

Using this, show

$$F_\omega(\phi_t) = -\frac{1}{t}\int_0^t (I_\omega - J_\omega)(\phi_s)\,ds - \log\left(\frac{1}{V}\int_X e^{h_\omega - \phi_t}\omega^n\right)$$

If $t = 1$, is zero! So

$$F_\omega(\phi_1) = -\int_0^1 (I_\omega - J_\omega)(\phi_s)\,ds$$

$$= -F_{\omega_{KE}}(\phi)$$

$$\Rightarrow \boxed{F_{KE}(\phi) = \int_0^1 (I_\omega - J_\omega)(\phi_t)\,dt}$$ Thing 2

Prove Thing 1

$$(MA) \Rightarrow \quad \omega_{\phi_t}^n = e^{h_\omega - t\phi_t}\,\omega^n$$

$$\frac{d}{dt}(\omega_{\phi_t}^n) = \Delta_t \dot{\phi}_t\, \omega_{\phi_t}\, \omega_{\phi_t}^n$$

Computations $\Rightarrow$ $\Delta_t \dot{\phi}_t = (-t\dot{\phi}_t - \phi_t)$

Now: $$\frac{d}{dt}(I_\omega - J_\omega)(\phi_t) = -\frac{1}{V}\int_X \phi_t \frac{d}{dt}(\omega_{\phi_t}^n)$$

$$= -\frac{1}{V}\int_X \phi_t(\Delta_t \dot{\phi}_t)\,\omega_{\phi_t}^n$$

$$\frac{d}{dt}(I_\omega - J_\omega)(\phi_t) = -\frac{1}{V}\int_X \phi_t(-t\dot{\phi} - \phi_t)\,\omega_{\phi_t}^n$$

$$= \frac{1}{V}\int_X \phi_t^2\, \omega_{\phi_t}^2 + \frac{t}{V}\int_X \phi_t \dot{\phi}_t\, \omega_{\phi_t}^n$$

$$\geq \frac{t}{V}\int_X \phi_t \dot{\phi}_t\, \omega_{\phi_t}^n$$

$$\geq \frac{t}{V}\int_V (-\Delta_t - t)\dot{\phi}_t \dot{\phi}_t \,\omega_{\phi_t}^n$$

Claim

$$= \frac{t}{V}\int_V ((-\Delta_t - t)\dot{\phi}, \dot{\phi})_{L^2_t} \geq 0$$

<u>Claim</u>: $-\Delta_t f = \lambda f \Rightarrow \lambda \geq t$

<u>Pf</u>: Use (MA)

$$\omega + \frac{i}{2} t \partial\bar{\partial} \phi_t = Ric(\omega_{\phi_t})$$

$$\omega + t(\omega_{\phi_t} - \omega) = Ric(\omega_{\phi_t})$$

$$t\omega_{\phi_t} + (1-t)\omega = Ric(\omega_{\phi_t})$$

$$t\omega_{\phi_t} \leq Ric(\phi_t)$$

$$\nabla^j f = g^{j\bar{k}} \nabla_{\bar{k}} f = -\frac{1}{\lambda} g^{j\bar{k}} \nabla_{\bar{k}} g^{l\bar{m}} \nabla_l \nabla_{\bar{m}} f$$

$$= -\frac{1}{\lambda} \Delta \Delta^j f + \frac{1}{\lambda}(g^{j\bar{k}} g^{l\bar{p}} R_{\bar{k}l} \nabla_{\bar{p}} f)$$

$$\nabla^j f \geq -\frac{1}{\lambda} \Delta\Delta^j f + \frac{t}{\lambda}\nabla^j f$$

$$\Rightarrow 0 \geq \frac{1}{\lambda}(\Delta \nabla^j f, \nabla^j f)_{L^2} \geq \left(\frac{t}{\lambda} - 1\right)|\nabla^j f|^2_{L^2}$$

Thing 3

$$\left| J_\omega(\phi_t) - J_\omega(\phi_0) \right| \leq \operatorname{osc}(\phi_1 - \phi_0), \quad \forall \phi_1, \phi_0$$

Pf: uses cocycle?

$$\cancel{F_\omega^\circ(\phi_1) - F_\omega^\circ}$$

$$F_\omega^\circ = -\frac{1}{V}\int_0$$

Thing 4 : $\left| (I_\omega - J_\omega)(\phi_1) - (I_\omega - J_\omega)(\phi_0) \right| \leq n \operatorname{osc}(\phi_1 - \phi_0)$

No proof.

Try to prove $(*)$.

$$F_{\omega_{KE}}(\phi) = \int_0^1 (I_\omega - J_\omega)(\phi_t)\, dt \qquad I_\omega \geq \frac{n+1}{n} J_\omega$$

$$\geq (1-t)(I_\omega - J_\omega)(\phi_t)$$

$$\geq \frac{1}{n}(1-t) J_\omega(\phi_t)$$

$$\geq \frac{1}{n}(1-t) J_\omega(\phi_1) - \frac{1}{n}(1-t)\,\mathrm{osc}(\phi_t - \phi_1)$$

$$\boxed{F_{\omega_{KE}}(\phi) \geq \frac{1}{n^2}(1-t) J_{\omega_{KE}}(\phi) - \frac{1}{n}(1-t)\,\mathrm{osc}(\phi_t - \phi_1)} \quad (\text{***})$$

Want to chang (MA):

$$(\omega + \tfrac{i}{2}\partial\bar\partial\phi_t)^n = e^{h_\omega - t\phi_t}\omega^n$$

$$\Rightarrow -\mathrm{Ric}(\omega_{\phi_t}) = -\omega - \tfrac{i}{2} t\,\partial\bar\partial\phi_t$$

$$-\omega = -\omega_{KE} + \tfrac{i}{2}\partial\bar\partial\phi_1$$

$$= -\mathrm{Ric}(\omega_{KE}) + \tfrac{i}{2}\partial\bar\partial\phi_1$$

$$= \tfrac{i}{2}\partial\bar\partial\log(\omega_{KE})^n + \tfrac{i}{2}\partial\bar\partial\phi_1$$

(MA)'

$$\Rightarrow \boxed{(t-1)\phi_t = \log\left(\frac{\omega_{KE}^n}{\omega_{KE} - \frac{i}{2}\partial\bar\partial(\phi_1 - \phi_t)^n}\right) + (\phi_1 - \phi_t)}$$

$$\tfrac{i}{2}\partial\bar\partial(t-1)\phi_t = -\omega_{KE} + \tfrac{i}{2}\partial\bar\partial(\phi_1 - \phi_t) + \mathrm{Ric}(\omega_{\phi_t})$$

$$= -(\omega_{KE} + \tfrac{i}{2}\partial\bar\partial\phi) - \tfrac{i}{2}\partial\bar\partial\phi_t + \mathrm{Ric}(\omega_{\phi_t})$$

$$= \mathrm{Ric}(\omega_{\phi_t}) - \omega_{\phi_t}$$

$$\int_X \omega_{\phi_t}^n = \int_X e^{(h_{\omega_{\phi_t}})} \omega_{\phi_t}^n$$

$$= \int_X \left(e^{(t-1)\phi_t + c_t}\right) \omega_{\phi_t}^n$$

$$|c_t| \le (1-t)\|\phi_t\|_{C^0}$$

$$\boxed{\|h_{\omega_{\phi_t}}\|_{C^0} \le 2(1-t)\|\phi_t\|_{C^0}}$$

$\rho_1 - \phi_t$?

$$(MA)' \qquad \log \frac{\omega_{KE}^n}{(\omega_{KE} + \frac{i}{2}\partial\bar{\partial}\psi)^n} + \psi = h$$

Linearize at $\psi = 0$:

$$\delta\psi \longmapsto (\Delta_{KE} + 1)\,\delta\psi$$

Green's Function gives inverse off kernel which is empty!

Bochner Kodaira

No holo VF $\Rightarrow$ no kernel (Tristan)

IFT : $\|h\|_{\mathcal{C}^{0,k}} < \varepsilon(\omega_{KE}) \Rightarrow \|\psi\|_{\mathcal{C}^{0,k}} \le C(\omega_{KE})\|h\|_{\mathcal{C}^{0,k}}$

$L^p_s \qquad\qquad L^p_s \qquad\qquad L^p_s$

see box prev page

OSC

$$(1-t)\|\phi_t\|_{\mathcal{C}^0} \le \varepsilon \Rightarrow \|\phi_1 - \phi_t\|_{\mathcal{C}^0} \le C(1-t)\|\phi_t\|_{\mathcal{C}^0}$$

Lemma 1: $\exists D$ s.t. $\|\phi_1 - \phi_t\|_{\mathcal{C}^0} \le 100(1-t)\|\phi_t\|_{\mathcal{C}^0} + 1$,

for $t \in [t_0, 1]$, where t_0 def by:

- $D = (1-t_0)^{1-\alpha}\left(1 + Q(1-t_0)\|\phi_t\|_{\mathcal{C}^0}\right)^{\alpha}$

(if t_0 goes bigger D goes smaller) $\qquad \alpha > \frac{1}{2}$

α is something...

Pf: IFT + KRF... come back later?

$$D = (1-t_0)^{1-\alpha}\left(1 + 2(1-t_0)\|\phi_{t_0}\|_{\mathcal{C}^0}\right)^{\alpha}$$

$$\geq 2^{\alpha}(1-t_0)\|\phi_{t_0}\|^{\alpha}_{\mathcal{C}^0}$$

$$\Rightarrow (1-t_0)^2\|\phi_{t_0}\|_{\mathcal{C}^0} \leq D^{1/\alpha}$$

(last page)

$$(\ast\ast\ast\ast) \Rightarrow (1-t_0)\|\phi_1 - \phi_{t_0}\|_{\mathcal{C}^0} \leq 100(1-t_0)^2\|\phi_{t_0}\|_{\mathcal{C}^0} + 1$$

$$\leq C_1$$

$$(1-t_0) < \delta$$

$$\Rightarrow (1-t_0)\|\phi_{t_0}\| \geq \frac{1}{2}$$

some computation n' shit give:

$$1 - t_0 \geq \frac{C_2}{\|\phi_{t_0}\|^{\alpha}_{\mathcal{C}^0}}$$

So $(\ast\ast\ast)$ becomes

$$F_{\omega_{KE}}(\phi) \geq \frac{C_2}{n^2}\frac{J_{\omega_{KE}}(\phi)}{\|\phi_{t_0}\|^{\alpha}_{\mathcal{C}^0}} - \frac{2}{n}C_1$$

$$\|\phi_{t_0}\|_{\mathcal{C}^0} \leq 2\|\phi_1\|_{\mathcal{C}^0} + 2$$

$$\|\phi_t\| - \|\phi_1\| \leq \|\phi_t - \phi_1\|_{\mathcal{C}^0}$$

$$\leq 100(1-t)\|\phi_t\| + 1$$

$$\Rightarrow \quad F_{\omega_{KE}}(\phi) \geq \frac{c_2}{n^2} \frac{J_{\omega_{KE}}(\phi)}{(2\,osc(\phi)+2)^{\alpha}} - \frac{2}{n} C_1$$

$$osc(\phi) \leq \varepsilon K \left(J(\phi) + 1 \right)$$

This gives Step 1

<u>Step 2</u> : We want

$$F_{\omega_{KE}}(\phi) \geq A\, J_{\omega_{KE}}(\phi) - B$$

<u>Fact 1</u> : $osc(\phi_t - \phi_1) \leq K \left(J_{\omega_{KE}}(\phi_t - \phi_1) + 1 \right)$

<u>Fact 2</u> : $\exists\, C, m$ s.t.

$$J_{KE}(\phi_t - \phi_1) \leq C \quad w/ \ 1 - t \geq m$$

Fact 1 + Fact 2 + ($\cancel{**}$) gives Thm 1

Proof (Fact 1)

$$\mathrm{Ric}(\omega_{\phi_t}) \geq t\,\omega_{\phi_t}$$

$$\geq \frac{1}{2}\,\omega_{\phi_t}, \quad t \in [\tfrac{1}{2}, 1]$$

$\Rightarrow$ Green's Function bdd below uniformly

$$\omega_{KE} + \frac{i}{2}\partial\bar{\partial}(\phi_t - \phi_1) = \omega_{\phi_t} > 0$$

(trace ω_{KE}) $$n \geq -\Delta_{KE}(\phi_t - \phi_1)$$

Same thing (trace ω_{ϕ_t}) gives

Green's function stuff

$$-n \leq -\Delta_t(\phi_t - \phi_1)$$

$$\Rightarrow \phi_t(x) - \phi_1(x) \leq \frac{1}{V}\int_X (\phi_t - \phi_1)\,\omega_{KE}^n + C_0$$

$$\phi_t(y) - \phi_1(y) \geq \frac{1}{V}\int_X (\phi_t - \phi_1)\,\omega_{\phi_t}^n - C_0$$

$$\Rightarrow \quad \operatorname{osc}(\phi_t - \phi_1) \le \frac{1}{V}\int_X (\phi_t - \phi_1)(\omega_{KE}^n - \omega_{\phi_t}^n) + 2c_0$$

$$= I_{\omega_{KE}}(\phi_t - \phi_1) + 2c_0$$

$$\le K\left(J_{\omega_{KE}}(\phi_t - \phi_1) + 1\right) \quad \boxed{\text{KE}}$$

<u>Proof (Fact 2)</u> $\qquad$ <u>cocycle</u>

$$F_{\omega_{KE}}(\phi_t - \phi_1) = F_\omega(\phi_t) - F_\omega(\phi_1)$$

$$\text{thing } 2 = -\frac{1}{t}\int_0^t (I_\omega - J_\omega)(\phi_s)\,ds \quad A$$

$$B \quad + \int_0^1 (I_\omega - J_\omega)(\phi_s)\,ds - \log\left(\frac{1}{V}\int_X e^{(t-1)\phi_t}\omega_{\phi_t}^n\right)$$

Jensen

$$\le A + B + \frac{(1-t)}{V}\int_X \phi_t\,\omega_{\phi_t}^n$$

lose $\frac{1}{t}$

$$\le A + B - (1-t)(I_\omega - J_\omega)(\phi_t) + \frac{(1-t)}{t}\int_0^t (I_\omega - J_\omega)\phi_s\,ds$$

$$= \int_t^1 (I_\omega - J_\omega)(\phi_s)\,ds - (1-t)(I_\omega - J_\omega)(\phi_t)$$

$$\leq (1-t)\left[(I\omega - J\omega)(\phi_1) - (I\omega - J\omega)(\phi_t)\right]$$

(Thing 4)

$$\Rightarrow \leq n(1-t)\,\mathrm{osc}(\phi_t - \phi_1)$$

(Fact 1)

$$\leq n(1-t)K\left(J_{\omega_{KE}}(\phi_t - \phi_1) + 1\right)$$

Also have from step 1:

$$A_\gamma J_{\omega_{KE}}(\phi_t - \phi_1)^\gamma - B_\gamma$$

$$\leq n(1-t)K\left(J_{KE}(\phi_t - \phi_1) + 1\right)$$

$$\Rightarrow J_{\omega_{KE}}(\phi_t - \phi_1)^\gamma \left(A_\gamma - nK(1-t) J_{\omega_{KE}}(\phi_t - \phi_1)^{1-\gamma}\right)$$

$$\leq n(1-t)K + B_\gamma$$

Then go away from 1 [| 1

and then combine w/ (***) to be done.