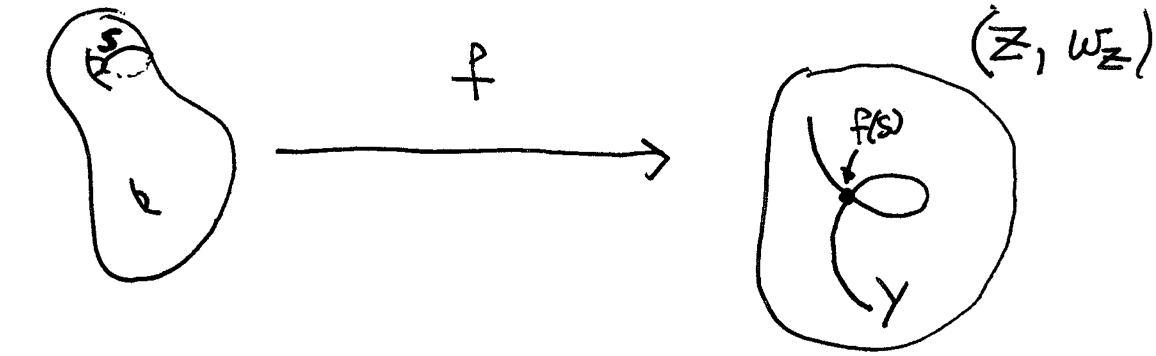


(X, ω_X) $\qquad\qquad$ $Y = f(X)$

cpt Kähler $\qquad\qquad$ $\dim_{\mathbb{C}} Y = m < n$

$\dim_{\mathbb{C}} X = n$, $c_1(X) = 0$

$$\omega_Y := \omega_Z\big|_Y$$

Define $\quad \omega_0 := f^*\omega_Z$ degenerate form on X (zero in fiber direction)

$S \subset X$ is a proper subvariety st. $Y \setminus f(S)$ is smooth

and $\quad f : X \setminus S \longrightarrow Y \setminus f(S)$ is a smooth submersion

Note $\quad [\omega_0]$ is in boundry of Kähler cone,

since $\quad [\omega_0] + t[\omega_X]$ is a Kähler class $\forall t \in (0,1]$

By Yau's thm, in each class we have

$\tilde{\omega}_t$ unique Ricci flat Kähler metric.

Study what happens as $t \longrightarrow 0$

Note if $m_{?} \le K \le n$ we have

$\omega_0^K \wedge \omega_X^{n-K} = 0$, and $\omega_0^m \wedge \omega_X^{n-m} = H\,\omega_X^n$

H is smooth, non-negative vanishes on S s.t. $H^{-\gamma}$ is L^1 for some $\gamma > 0$ (since locally H is comparable to sums of squares of holó (minors of Jacobian of f))

$\int_X \omega_0^m \wedge \omega_X^{n-m} > 0$ well defined

Also have defining function for $f(S)$.

Let $\mathcal{I}$ be ideal sheaf for $f(S)$

cover Z by U_k so $\mathcal{I}$ generated by $\{h_{k,1}, \ldots, h_{k,N_k}\}$

Fix η_k POU set

$$\sigma = \sum_K \eta_k \left(|h_{k,1}|^2 + \cdots + |h_{k,N_k}|^2 \right)$$

$\sigma \le C$, $0 \le i\partial\sigma\bar\partial \le C\omega_Z$, $-C\omega_Z \le i\partial\bar\partial\sigma \le C\omega_Z$

furthermore H & $f^*\sigma$ are both locally comparable to sums of holó squared same vanishing locus

$\Rightarrow$ $\sigma(y)^\lambda \le C \inf_{X_y} H$

Lets define the MA equation

Reference form $\omega_t := \omega_0 + t\,\omega_X$, $[\tilde{\omega}_t] = [\omega_t]$

both 0, $Ric(\omega_X) \in c_1(X)$

$\Rightarrow$ $Ric(\omega_X) = i\partial\bar{\partial}E$ normalized $\int_X e^E \omega_X^n = \int_X \omega_1$

Consider the following MA equation

$$\underset{\substack{\shortparallel \\ (\omega_t + i\partial\bar{\partial}\phi_t)^n}}{\tilde{\omega}_t^n} = a_t\, e^E \omega_X^n \quad \text{where } a_t = \frac{\int_X \omega_t^n}{\int_X \omega_1^n}$$

Note $-\partial\bar{\partial}\log$ of both sides you have

$$Ric(\tilde{\omega}_t) = -\partial\bar{\partial}E + Ric(\omega_X) = 0$$

Note for small t

$$a_t \approx \binom{n}{m} \frac{\int_X \omega_0^m \wedge \omega_X^{n-m}}{\int_X \omega_1^n} t^{n-m} + O(t^{n-m+1})$$

thus

$$(\omega_t + i\partial\bar{\partial}\phi_t)^n = c_t\, t^{n-m} e^E \omega_X^n$$

does not vanish at 0

Thm (PP, EGZ 2)

$\exists\, C$ depending on $(X, E, \omega_X, \omega_0)$ st $\forall t \in (0,1]$

$$\|\varphi_t\|_{L^\infty} \leq C$$

Goals
- Higher order estimates ~~convergence~~
- Convergence (in what norm and to what limit?)

first, estimates

$\boxed{\text{Thm}}$ 2.2 $\exists$ A, B, C st on $X \setminus S$ we have

$$\frac{t}{C\underbrace{e^{Ae^{B_\sigma^{-\lambda}}}}_{Q}}\,\omega_X \leq \tilde{\omega}_t \leq C\underbrace{e^{Ae^{B_\sigma^{-\lambda}}}}_{Q}\,\omega_X \qquad \boxed{2.9}$$

$\Rightarrow$ $\Delta_{\omega_X} \varphi_t$ bdd on cpt sets of $X \setminus S$

Now given $y \in Y \setminus f(S)$ denote: $X_y := f^{-1}(y)$

$\omega_y := \omega_X|_{X_y}$ $\quad \tilde{\omega}_y := \tilde{\omega}_t|_{X_y}$

Thm 2.3 fiber collapsing (

on X_y, $\forall t \in [0,1]$

$\boxed{2.10}$ $$\frac{t}{Q}\,\omega_y \le \tilde{\omega}_y \le t\,Q\,\omega_y$$

$$|\nabla \tilde{\omega}_y|^2_{\omega_y} \le t^{1/2}\,Q$$

$\Rightarrow$ $\tilde{\omega}_t \to 0$ in C^1 as $t \to 0$ uniformly

---

Thm 2.4 $\boxed{\text{Convergence}}$

As $t \to 0$ on $X \setminus S$ $\tilde{\omega}_t \to f^*\omega$ weakly as currents

potentials $C^{1,\beta}_{loc}$

where ω smooth Kähler metric on $Y \setminus f(S)$

$$\mathrm{Ric}(\omega) = \omega_{WP}$$ ← Weil-Petersson metric from moduli space of CY fibers

we have

$$g_x \ , \quad g_t = g_0 + t g_x$$

$$\tilde{g}_t = g_t + \partial\bar{\partial}\phi_t$$

$$0 < g_x^{-1}\tilde{g}_t = \underbrace{g_x^{-1} g_0}_{\text{fixed}} + tn + \Delta_x \phi_t$$

always have uniform lower bound in t.

<u>lemma</u> $\exists$ C uniform s.t. $\tilde{g}_t^{-1} g_0 \leq C$

Apply Yau's Schwarts

$$\Delta_{\tilde{g}_t} \log \tilde{g}_t^{-1} g_0 \geq -A\, \tilde{g}_t^{-1} g_0$$

also $\Delta_{\tilde{g}_t} \varphi_t = n - \tilde{g}_t g_t \leq n - \tilde{g}_t g_0$

$$\Rightarrow \Delta_{\tilde{g}_t}\left(\log \tilde{g}_t^{-1} g_0 - (A+1)\varphi_t\right) \geq \tilde{g}_t^{-1} g_0 - n(A+1)$$

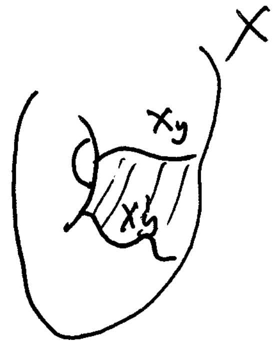
max principal + <u>EGZ</u> L^∞ for ϕ_t

$\Rightarrow$ done

Thm (Federer)

$X \backslash S$ ← Kähler manifold

X_y submanifold



consider all

real cobordism-~~deformations~~ to another 2^n dim submanifolds

If X_y is ~~Kähler~~ a complex submanifold

has minimal volume ~~for~~ over all X'_y. furthermore

If $vol(X_y) = vol(X'_y) \Rightarrow X'_y$ is also cplx

Lemma Uniform Sobolev constant on X_y

$\forall t \in (0,1]$ $\exists y \in Y \backslash f(S)$ $\forall u \in C^\infty(X_y)$

$$\Rightarrow \left(\int_{X_y} |u|^{\frac{2(n-m)}{n-m-1}} \omega_y^{n-m}\right)^{\frac{n-m-1}{n-m}} \le C \int_{X_y} \left(|\nabla u|^2_{\omega_y} + |u|^2\right) \omega_y^{n-m}$$

proof (mean curvature zero) isometrically

embed in $\mathbb{R}^N$ then mean curvature bounded

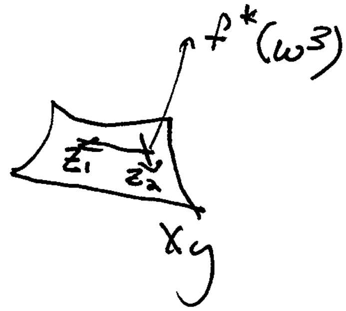
$X_y \to X \to \mathbb{R}^N$, so that gives uniform sobolev bd.

This also implies a diameter bound

$$\exists C \quad st \quad \mathrm{diam}(X_y, \omega_y) \leq C$$

<u>Next step</u> Poincaré inequality

Constant blows up like $\frac{1}{H}$



~~first estimate $Ric(\omega_y)$~~

coords on X_y $\quad z^1, \ldots, z^{n-m}$

$\omega^{n-m+1}, \ldots, \omega^n$ near $y \in Y \setminus f(S)$, so $z^{n-m+i} = f^*(\omega^{n-m+i})$

$$Ric(\omega_y) = -i\partial\bar\partial \log \frac{\omega_y^{n-m}}{dz^1 \wedge \cdots \wedge d\bar z^{n-m}}$$

$$= -i\partial\bar\partial \log \frac{\omega_X^{n-m} \wedge \omega_0^m}{dz^1 \wedge \cdots \wedge d\bar z^n}$$

$$= -i\partial\bar\partial \log H - i\partial\bar\partial \log \frac{\omega_X^n}{dz^1 \wedge \cdots \wedge d\bar z^n}$$

derivatives in fiber directions

$$\geq -\frac{i\partial\bar\partial H}{H} + Ric(\omega_X)|_{X_y}$$

$$\geq -\left(\frac{C}{H} + C\right)\omega_y \geq -\frac{C}{H}\omega_y \sim -C\sigma^{-\lambda}$$

diameter bound + Ricci lower bound

$\Rightarrow \quad \forall y \in Y \setminus f(S) \quad u \in C^\infty(X_y) \quad$ with $\int_{X_y} u\, \omega_y^{n-m} = 0$

$$\Rightarrow \int_{X_y} |u|^2 \omega_y^{n-m} \leq C e^{B_0 - \lambda} \int_X |\nabla u|^2_{\omega_y} \omega_y^{n-m}$$

Recall $\tilde{\omega}_y := \tilde{\omega}_t|_{X_y}$

Volume form estimate
$$\frac{\tilde{\omega}_y^{\,n-m}}{\omega_y^{\,n-m}} = \frac{\tilde{\omega}_y^{\,n-m}\wedge\omega_0^{\,m}}{\omega_X^{\,n-m}\wedge\omega_0^{\,m}} = \frac{\tilde{\omega}_t^{\,n-m}\wedge\omega_0^{\,m}}{\tilde{\omega}_t^{\,n}}\cdot\frac{\tilde{\omega}_t^{\,n}}{H\,\omega_X^{\,n}}$$

$$\leq \left(\frac{\tilde{\omega}_t^{\,n-1}\wedge\omega_0}{\tilde{\omega}_t^{\,n}}\right)^m \frac{c_t\, t^{n-m} e^{H}}{H}$$

$$= \left(\tilde{g}_t^{-1} g_0\right) \frac{c_t\, t^{n-m} e^{\epsilon}}{H} \leq \frac{C t^{n-m}}{\sigma^{\lambda}}$$

~~Proof of Thm 2.2~~

~~Apply max principle to~~

$$\cancel{K = e^{-B\sigma^{-\lambda}}\left(\log (g_X^{-1}\tilde{g}_t) - \frac{A}{t}(\varphi_t - \underline{\varphi_t}\right)}$$

now $\tilde{\omega}_y = (\omega_0 + t\omega_X + i\partial\bar{\partial}\varphi_t)|_{X_y} = t\omega_y + (i\partial\bar{\partial}\varphi_t)|_X$

define $\underline{\varphi_t}(y) \in C^\infty(Y\setminus f(S))$

$$\underline{\varphi_t}(y) = \int_{X_y} \varphi_t\, \omega_y^{\,n-m}$$

"integration along fibers"

define $\psi = \frac{1}{t}(\varphi_t - \underline{\varphi_t})$, thus $\int_{X_y} \psi \, \omega_y^{n-m} = 0$

$$\Rightarrow \underbrace{(\omega_y + i\partial\bar\partial \psi)^{n-m}}_{} = \frac{\tilde{\omega}_y^{n-m}}{t^{n-m}} \leq \frac{C}{\sigma^\lambda} \omega_y^{n-m}$$

Apply Yau's L^∞ estimate (Note poincare inequality constant)

$$\sup_{X_y} |\varphi_t - \underline{\varphi_t}| = t \sup_{X_y} |\psi| \leq t \, C e^{B\sigma(y)^{-\lambda}}$$

To proove Thm $\underline{2.2}$

Apply max principle to

$$K = e^{-B\sigma^{-\lambda}} \left(\log (g_0^{-1} \tilde{g}_t) - \frac{A}{t}(\varphi_t - \underline{\varphi_t}) \right)$$

got ~~huge~~ mess

For the left inequality

$$K_1 = e^{-B\sigma^{-\lambda}} \left(\log (t \, \tilde{g}_t^{-1} g_X) - \frac{A}{t}(\varphi_t - \underline{\varphi_t}) \right)$$

less of a mess

Now we show How 2A => 2.10 LHS follow immediately.

$$\Longrightarrow \quad (\tilde{g}_y^{-1} g) \le \frac{C}{t} \, e^{Ce^{B_\sigma - \lambda}} \quad *$$

Now for any Kähler metric $g^{-1}\tilde{g} \le (\tilde{g}^{-1} g) \frac{\tilde{\omega}^n}{\omega^n}$

so

$$g_y^{-1} \tilde{g}_y \le (\tilde{g}_y^{-1} g)^{n-m-1} \frac{\tilde{\omega}_y^{\,n-m}}{\omega_y^{\,n-m}}$$

volume form estimate

$$\le (\tilde{g}_y^{-1} g)^{n-m-1} \frac{C \cancel{t^{n-m}}}{\sigma\lambda}$$

$*$

$$\le \frac{\left(C e^{Ce^{B_\sigma - \lambda}}\right)^{n-m-1}}{\cancel{t^{n-m-1}}}$$

$$\le t \frac{Ce^{Ce^{B_\sigma - \lambda}}}{\sigma\lambda} \le t C e^{Ce^{B_\sigma - \lambda}}$$

proving $\boxed{2.10}$

---

for 2.11 $\quad |\nabla \tilde{\omega}_y|^2_{\omega_y} \le t^{1/2}$. Q

Let $\quad S = |\nabla \tilde{\omega}_t|^2_{\tilde{\omega}_t}$

Apply Max principle to

$$K_2 = e^{-Ae^{B_0 - \lambda}} \left(S + C \frac{e^{3C_0 e^{B_0 - \lambda}}}{t^{5/2}} \, g_x^{-1} \tilde{g}_t \right)$$

~~If at max $K_2 \geq \frac{C}{t^{5/2}}$ on X_y $\Rightarrow \lambda$ $g_y^{-1} \tilde{g}_y \leq t\, Ce$~~ . Huge mess ... again!

$$K_2 \leq \frac{C}{t^{5/2}} \Rightarrow S \leq \frac{C e^{Ae^{B_0 - \lambda}}}{t^{5/2}}$$

Of course for S we took the norm w.r.t. $\tilde{\omega}_t$

And Thm is for ω_y (no tilde)

So... at any given point in X_y, assume ω_X is the identity and $\tilde{\omega}_t$ is diagonal with entries λ_i, $1 \leq i \leq n$, so $n-m$ directions are tangent to X_y

Then by $\boxed{2.10}$ we have

$$\lambda_i \leq t\, C e^{C_0 e^{B_0 - \lambda}} \qquad 1 \leq i \leq n-m$$

Thus

$$\sum_{i,j,k=1}^{n-m} \frac{1}{\lambda_i \lambda_j \lambda_k} |\varphi_{i\bar{j}k}| \leq \sum_{i,j,k=1}^{n-m} \frac{1}{\lambda_i \lambda_j \lambda_k} \left| g^0_{\bar{j}i,k} + \varphi_{i\bar{j}k} \right|^2 = S \leq \frac{Q}{t^{5/2}}$$

Convergence Thm 2.4

Recall $\tilde{\omega}_t \longrightarrow f^*\omega$ weakly as currents

potentials in $C^{1,\beta}_{loc}$

lets describe ω first

Rebee $\omega_1 = \omega_0 + \omega_X$, $\tilde{\omega}_1$ Ricci flat

$\Omega := \tilde{\omega}_1^n$

Now $c_1(X_y) = 0 \Rightarrow Ric(\omega_y) = i\partial\bar{\partial} F_y$

with $\int_{X_y} (e^{F_y} - 1)\, \omega_y^{n-m} = 0$

By Yau's Thm $\exists$ $\omega_{SF,y}$ ↙ Ricci flat on X_y in $[\omega_y]$

s.t. $\omega_{SF,y}^{n-m} = e^{F_y}\, \omega_y^{n-m}$

$\omega_{SF,y} = \omega_y + i\partial\bar{\partial}\zeta_y$ ← varies smoothly in y (since ω_y does)

defines $\zeta \in C^\infty(X \setminus S)$

$\omega_{SF} = \omega_X + i\partial\bar{\partial}\zeta$ ← not nonnegative (Kähler only in fiber directions) but $\omega_{SF}^{n-m} \wedge \omega_0^m$ strictly possitive

define $F = \dfrac{\Omega}{\omega_{SF}^{n-m} \wedge \omega_0^m}$

Claim F constant on each fiber

proof on each Fiber you can show (derivatives in fiber-direction only

$i\partial\bar\partial \log F = -Ric(\tilde{\omega}_1) + Ric(\omega_{SF,y}) = 0$

On $Y \setminus f(S)$ $\quad F = \dfrac{f_* \Omega}{\omega_Y^m} \quad + \quad \int_Y F \omega_Y^m = \int_X \Omega = \int_X \omega_1^n$ finite

in fact $\int_Y F^{1+\varepsilon} \omega_Y^m$ finite for some ε.

$\boxed{4.3}$ $(\omega_Y + i\partial\bar\partial\varphi)^m = \dfrac{\int_X \omega_0^m \wedge \omega_X^{n-m}}{\int_X \omega_1^n} F \omega_Y^m$

Solve by Kołodziej and EGZ1

$\varphi \in L^\infty(Y)$, smooth on $Y \setminus f(S)$

Let $\omega = \omega_Y + i\partial\bar\partial\phi$

$\underline{\hspace{6cm}}$ ⇐ this guy is the limit

Describe Weil-Petersson metric

curvature form of a psuedonorm on relative canonical ~~bun~~ line bundle

$$f_*\left(\Omega_{X/Y}^{n-m}\right)^{\otimes k}$$

where k is choosen so $K_{X_y}^{\otimes k}$ trivial $\forall y$

If Ψ_y nonvanishing section

$$|\Psi_y|^2_{h_{WP}} = \int_{X_y} \left(\Psi_y \wedge \overline{\Psi_y}\right)^{1/k}$$

not metric, just psuedonorm

$$\omega_{WP} = i\partial\bar\partial \log |\Psi_y|^2_{h_{WP}}$$

Prop On $Y \setminus f(S)$, $\mathrm{Ric}(\omega) = \omega_{WP}$

Proof of convergence

$$\varphi_t \to \psi \quad \text{in} \quad C^{1,\beta}_{loc}$$

Note $[\tilde{\omega}_t]$ bounded, weak compactness of currents

implies subsequence $\tilde{\omega}_{t_i} \xrightarrow{\text{weakly}} \hat{\omega}$ ← positive (1,1) current

$\hat{\omega} = \omega_0 + i\partial\bar\partial\hat{\varphi}$, then by weak convergence

$\varphi_{t_i} \to \hat{\varphi}$ in L^1.

By EGZ 2 $\qquad \hat{\varphi} \in L^\infty$

Furthermore, restrict to any fiber, see that $\omega_0|_{X_y} = 0$

so

$$i\partial\bar\partial\hat{\varphi}\Big|_{X_y} \geq 0$$

$\Rightarrow$ So by Max principle constant on each fiber

we show $\hat{\varphi}$ solves 4.3, so by uniqueness $\hat{\varphi} = \psi$

Also Thm 2.2 implies

$$-C\omega_X \leq i\partial\bar\partial\varphi_t \leq C\omega_X \Rightarrow \text{laplacian bound}$$

$\Rightarrow$ further subsequence convergenes in $C^{1,\beta}(K)$