KR - Solitons on Fano Toric Varieties

Want to solve:

$$\boxed{Ric(\omega_g) - \omega_g = L_X(\omega_g)}$$

for (X, g).

① Need to find X.
② Need to find g.

Fano implies:

$$\begin{cases} Ric(\omega_g) - \omega_g = \frac{i}{2\pi}\partial\bar\partial h \\ \int_M e^h \omega_g^n = \int_M \omega_g^n \end{cases}$$

Now to define Futaki invariant.

Since $c_1 > 0$, $H^{0,1} = H^{1,0} = 0$. (???)

Thus if α is a closed $(0,1)$-form, it is $\bar\partial$-exact!

<u>Note:</u> If α is $(1,1)$ and X is holo, then $\bar\partial(\iota_X\alpha) = -\iota_X\bar\partial\alpha$.

$$\bar\partial(\iota_X\alpha) = \bar\partial\left(X^j\alpha_{\bar k j}\, d\bar z^k\right)$$

$$= X^j\,\partial_{\bar\ell}\alpha_{\bar k j}\, d\bar z^\ell \wedge d\bar z^k$$

$$= X \lrcorner\, \partial_{\bar\ell}\alpha_{\bar k j}\, dz^j \wedge d\bar z^\ell \wedge d\bar z^k$$

$$= -\iota_X\bar\partial\alpha$$

Thus if X holo,

$$\bar\partial(\iota_X\omega) = 0$$

Thus for any $X \in \eta(M)$ set:

$$\begin{cases} \iota_X \omega_g = \frac{i}{2\pi} \bar\partial \theta_X \\ \int_M e^{\theta_X} \omega_g^n = \int_M \omega_g^n \end{cases}$$

<u>Note:</u> θ_X depends on g! If (g, X) happens to be Ricci-Soliton, then

$$L_X \omega_g = d(\iota_X \omega_g) = \frac{i}{2\pi} \partial\bar\partial \theta_X$$

i.e. this means $h = \theta_X$!

Now define Futaki invariant:

$$\boxed{\mathcal{F}_X(v) = \int_M v(h - \theta_X) e^{\theta_X} \omega_g^n, \quad v \in \eta(M)}$$

$\mathcal{F}_X$ is ind of $\omega_g \in [\omega_g]$ (TZ2)!

Thus ~~the~~ first step to finding (X, g), is finding X s.t. $\mathcal{F}_X = 0$! For toric we can do this!

Toric

Def: (M^n, ω) is *toric* if $\exists$ action $(\mathbb{C}^*)^n \circlearrowright (M, \omega)$ by biholomorphic symplectomorphisms, and $\exists$ pt $p \in M$ s.t. $(\mathbb{C}^*)^n \cdot p$ is a dense orbit upon which $(\mathbb{C}^*)^n$ acts freely/transitively.

- e.g. $\mathbb{P}^n$, Hirzebruch, etc...

Note: If $f \in$ ~~Al~~ $\mathrm{Aut}(M)$ commutes w/ $(\mathbb{C}^*)^n$ then f determined by value at single point so much must actually be in $(\mathbb{C}^*)^n$.

I.e. the maximal torus of $\mathrm{Aut}(M)$ is actually $(\mathbb{C}^*)^n$. Also anything commuting w/ $(\mathbb{C}^*)^n$ is already in them (will use this).

<u>Def</u>: Let $\eta = \mathrm{Lie}(\mathrm{Aut}(M))$

$\eta_0 = \mathrm{Lie}((\mathbb{C}^*)^n)$

$\eta_0 \subset \eta$ is maximal <u>abelian</u>!

The Cartan Decomposition <u>says</u>:

$$\eta(M) = \eta_0(M) \oplus \sum_i \mathbb{C} v_i,$$

where v_i are representation of the roots

e.g. $\mathbb{P}^n \leftarrow$ acts. $\mathbb{P}GL(n+1)$

$$\begin{array}{c} \mathbb{P}GL(n+1) \\ \cup \\ (\mathbb{C}^*)^n \end{array}$$

For $\mathbb{P}GL(n+1)$ have

$$\eta_0 = \begin{bmatrix} * & & \\ & \ddots & \\ & & * \end{bmatrix} \qquad \eta = \begin{bmatrix} & & * \\ & \ddots & \\ 0 & & 0 \end{bmatrix}$$

For v_i have for $v \in \eta_0$.

$$\begin{aligned} \mathrm{ad}_v [v_i] &= [v, v_i] \\ &= \lambda_i(v) v_i \end{aligned}$$

where $\lambda_i \in \eta_0^*$. If $v \notin \eta_0$, $\exists w \in \eta_0$ s.t.

$$[w, v] \neq 0.$$

<u>Lemma</u>: M^n toric. Then $\exists X \in \eta_0(M)$ unique s.t. $\mathrm{im}\, X$ gen 1-param compact subgroup in $\mathrm{Aut}(M)$ <u>and</u>

$$F_X(v) = 0, \quad \forall v \in \eta(M).$$

Pf: Facts from [TZ2]. Under assumptions minus toric, $\exists!\ X \in \mathfrak{h}_r(M)$ s.t.

- $\operatorname{im}(X)$ gen compact subgroup of $\operatorname{Aut}(M)$
- $\mathcal{F}_X(v) \equiv 0$, $\forall v \in \mathfrak{h}_r$
- X in center of $\mathfrak{h}_r$ (!!!)
- $\mathcal{F}_X([v, v']) \equiv 0$, $\forall v \in \mathfrak{h}_r$, $v' \in \mathfrak{h}$. (!!!)

Where $\mathfrak{h}_r$ is some reductive (???) group which contains $\mathfrak{h}$ (!!!)

In our case: Since $X \in \operatorname{center}(\mathfrak{h}_r)$, $X \in \mathfrak{h}_0$.

Also, $\mathcal{F}_X(v) = 0$, $\forall v \in \mathfrak{h}_r$ so need to show $\mathcal{F}_X(v_i) = 0$, $\forall i$. But

$$\begin{aligned}\mathcal{F}_X(v_i) &= \lambda(v)^{-1}\mathcal{F}_X(\lambda(v)v_i)\\ &= \lambda(v)^{-1}\mathcal{F}_X([v, v_i])\\ &= 0\end{aligned}$$

For some $v \in \eta_0 \subset \eta_r$. $\boxed{}$

Back To Toric

On the free orbit $(\mathbb{C}^\times)^n \subset M$ have coordinates $(z_1, \dots, z_n)$ where action is standard. Thus η_0 given by:

$$\eta_0 = \operatorname{span}_{\mathbb{C}} \langle z_1 \frac{\partial}{\partial z_1}, \dots, z_n \frac{\partial}{\partial z_n} \rangle$$

Use exponential as a universal cover:

$$(w_1, \dots, w_n) \overset{\exp}{\longmapsto} (e^{w_1}, \dots, e^{w_n}) = (z_1, \dots, z_n)$$

Then we have

$$\exp_*\left(\frac{\partial}{\partial w_j}\right) = z_j \frac{\partial}{\partial z_j}$$

Thus our lemma tells us:

$$\boxed{X = \sum_i c_i \frac{\partial}{\partial w_i} = \sum_i c_i X_i}$$

Now looking back at our Futaki invariant we see that $\mathcal{F}_X$ is defined using a metric to get both ω_g^n (vol form) and $\mathcal{E}_X$!

For toric we have a nice metric to work with!

Moment Map

If (M^n, ω) is toric, there is always a moment map (def up to sc vector)

$$m : M \longrightarrow t^* \simeq \mathbb{R}^n$$

which is equivariant and whose image is a convex polytope $\Omega \subset \mathbb{R}^n$.

$$\Omega = co(p^{(0)}, \ldots, p^{(N)})$$

where $p^{(0)}, \ldots, p^{(N)}$ are vertices.

These vertices give an embedding

$$M \longrightarrow \mathbb{P}^N$$

E.g. $\mathbb{P}^n \ni [z_0 : \cdots : z_n] \mapsto \left(\frac{|z_1|^2}{|z_0|^2 + \cdots + |z_n|^2}, \cdots, \frac{|z_n|^2}{|z_0|^2 + \cdots + |z_n|^2} \right)$

This maps $\mathbb{P}^n \longrightarrow$ standard n-simplex $\subset \mathbb{R}^n$.

In this case, the embedding we get is identity.

In the case of $\mathbb{P}^n$ we have FS metric

$$\log(1 + |z_1|^2 + \cdots + |z_n|^2)$$

$$\parallel$$

$$\log(1 + e^{2x_1} + \cdots + e^{2x_n})$$

In general, we can define the metric by

$$u^0(x) = \log\left(\sum_{i=0}^{N} e^{\langle p^{(i)}, x \rangle} \right)$$

Where here I'm using coordinates

$$w_j = x_j + i\theta_j$$

on the pull back. A toric metric is just a convex function of variables x_i!

Now since $\frac{i}{2\pi}\partial\bar\partial u^\circ \in c_1(M)$, we have that

$$\omega_{g_0} + \frac{i\,\partial\bar\partial h}{2\pi} - \mathrm{Ric}(\omega_{g_0}) = 0$$

for a Ricci potential h. Thus have

$$\partial\bar\partial\left(u^\circ + h + \log\det u^\circ_{ij}\right) = 0 \text{ on } \mathbb{C}^n.$$

But then $u^\circ + h + \log\det u_{ij}$ analytic (??).

Since h bdd and since $e^{u^\circ}\det u^\circ_{ij}$ bdd (this is computation) and analytic. Thus for some $u^\circ \to u^\circ + C$ we have

$$\boxed{\det(u^\circ_{ij}) = e^{-h-u_0}}$$

Now if we go back to our definition of $\mathcal{E}_{X_i}$, we see $(\omega = \omega_{u^0})$

$$i_{X_i}\,\omega = \frac{i}{2\pi} u^0_{ij}$$
$$= \frac{i}{2\pi} \partial_j (u^0_i)$$

Thus we have

$$\mathcal{E}_{X_i} = \frac{\partial u^0}{\partial x_i} + b_i ,$$

for b_i cst so that

$$\left[\int_{\mathbb{R}^n} e^{\frac{\partial u^0}{\partial x_i} + b_i} \det(u^0_{ij}) = \int_{\mathbb{R}^n} \det(u^0_{ij}) \right]$$

Direct computation gives you:

$$\bar{\partial}(\Delta \mathcal{E}_v + \mathcal{E}_v + v(h)) = 0$$

for any $v \in \eta(M)$.

$$\theta_V = v^i \frac{\partial u}{\partial x^i} + v^i b_i$$

$$\bar\partial\left(\Delta\theta_v + \theta_v + v(h)\right) \overset{?}{=} 0$$

$$\partial_{\bar k}\left(u^{ij}\partial^2\left(v^\ell \frac{\partial u}{\partial x^\ell}\right) + v^\ell \frac{\partial u}{\partial x^\ell} + v^\ell \frac{\partial h}{\partial x^\ell}\right)$$

$j = \bar j$!

$$= \partial_{\bar k}\left(u^{ij}\partial_i\left(v^\ell u_{\ell j}\right)\right) + v^\ell u_{\ell \bar k} + v^\ell h_{\ell \bar k}$$

$$= \partial_{\bar k}\left(u^{ij}\partial_i\left(v^\ell u_{\ell j}\right)\right) + \cancel{v^\ell u_{\ell \bar k}} - \cancel{v^\ell u_{\ell \bar k}} + v^\ell\left(-\log\det u_{ab}\right)_{\ell \bar k}$$

$$= \partial_{\bar k}\left(u^{ij}\partial_i\left(v^\ell u_{\ell j}\right)\right) - \partial_{\bar k}\left(v^\ell u^{ab} u_{ab\ell}\right)$$

$$= \partial_{\bar k}\left[u^{ij} v^\ell u_{ij\ell} + u^{ij}\partial_i v^\ell u_{\ell j} - v^\ell u^{ab} u_{ab\ell}\right]$$

$$= \partial_{\bar k}\left[\delta_{i\ell}\,\partial_i v^\ell\right] = 0 \,!$$ (holomorphic) !

Thus we have then that

$$v(h) = c_v - \Delta \theta_v - \theta_v$$

for c_v uniquely determined by our various normalization conditions (θ_v only actually).

Now if we compute $\mathcal{F}_X(v)$ w/ u^0:

$$\mathcal{F}_X(v) = \int_M v(h - \theta_X) e^{\theta_X} \omega_{g^0}^n$$

$$= \int_M [c_v - \Delta\theta_v - \theta_v - v(\theta_X)] e^{\theta_X} \omega_{g^0}^n$$

$$= -\int_M (\theta_v - c_v) e^{\theta_X} \omega_{g^0}^n - \int_M (\Delta\theta_v - v(\theta_X)) e^{\theta_X} \omega_{g^0}^n$$

$$= -\int_M (\theta_v - c_v) e^{\theta_X} \omega_{g^0}^n$$

$$\int_M \left[\Delta \theta_v + v(\theta_X)\right] e^{\theta_X}\, \omega_{g_0}^n = 0 \ ?$$

$$\downarrow$$

$$\int_M u^{i\bar j} \frac{\partial^2 \left(v^\ell \frac{\partial u}{\partial x^\ell}\right)}{\partial z^i \partial \bar z^j}\, e^{\theta_X}\, \omega_{g_0}^n$$

$$= \int_M u^{i\bar j} \frac{\partial}{\partial z^i}\left(v^\ell u_{\ell \bar j}\right) e^{\theta_X}\, \omega_{g_0}^n$$

$$= -\int_M u^{i\bar j} v^\ell u_{\ell \bar j}\, \nabla_{\bar i}\, e^{\theta_X}\, \omega_{g_0}^n$$

$$= -\int_M \delta_{i\ell}\, v^\ell\, \partial_{\bar i}\, \theta_X\, e^{\theta_X}\, \omega_{g_0}^n$$

Now we want to show $b_i = c_i$!

First note that

$$\int_M (\ell_V - c_V) e^h \omega_{g^h}^n = -\int_M (V(h) + \Delta \ell_V) e^h \omega_{g^h}^n$$

$$= 0$$

by basically the same computation. However,

$$e^h = e^{-u_0^0} \det(u_{ij}^0)^{-1}$$

and since $\omega_{g^h}^n = \det(u_{ij}^0)$, ✓ const as well!

left side is just: $(V = X_i)$

$$\int_M (\ell_{X_i} - c_{X_i}) e^{-u_0^0} dx = \int_M \left(\frac{\partial u^0}{\partial x_i} + b_i - c_{X_i}\right) e^{-u^0} dx$$

$$= (b_i - c_{X_i}) \int_M e^{-u^0} dx$$

$$= 0.$$

$$\left(\int_M \frac{\partial u^0}{\partial x_i} e^{-u^0} dx = 0! \right)$$

Thus if we use our nice metric u^0, we see that

$$\boxed{\mathcal{E}_{X_i} = \frac{\partial u^0}{\partial X_i} + c_i}$$

Now to specify c_i better:

Lemma $c_1, \dots, c_n$ satisfy:

$$\int_{\Omega^*} y_i \exp\left\{\sum_{\ell=1}^{n} c_\ell y_\ell\right\} dy = 0, \quad i = 1, \dots, n$$

Pf: We already showed:

$$\mathfrak{J}_X(v) = -\int_M (\mathcal{E}_{X_i} - c_{X_i}) e^{\mathcal{E}_X} \overbrace{\det(u^0_{pq})}^{\omega_g^n}$$

$$= -\int_{\mathbb{R}^n} \frac{\partial u^0}{\partial x_i} e^{\sum c_\ell \frac{\partial u^0}{\partial x_\ell} + k} \det(u^0_{pq})$$

$$= -\int_{\Omega} y_i e^{\sum c_\ell y_\ell} e^k dy$$

$$= 0 . \qquad \blacksquare$$

Thus we've determined c_i uniquely by topological data (the polytope!)

Now to set ~~it~~ up the equation. We want to solve:

$$\boxed{\mathrm{Ric}(\omega_g) - \omega_g = \cancel{\tfrac{i}{2\pi}}\,\partial\bar\partial L_X(\omega_g)}$$

Now let $g = g^0 + \partial\bar\partial\varphi$. Then we have

$$\mathrm{Ric}(\omega_g) - \omega_g =$$

$$\mathrm{Ric}\left(\omega_{g^0} + \tfrac{i}{2\pi}\partial\bar\partial\varphi\right) - \omega_{g^0} - \tfrac{i}{2\pi}\partial\bar\partial\varphi = \overset{\frac{i}{2\pi}\partial\bar\partial\theta_X}{\overset{\shortparallel}{L_X(\omega_{g^0})}} + \overset{\frac{i}{2\pi}\partial\bar\partial X(\varphi)}{\overset{\shortparallel}{L_X\left(\tfrac{i}{2\pi}\partial\bar\partial\varphi\right)}}$$

$$\Rightarrow\ -\log\det\left(g^0_{\bar b a} + \varphi_{\bar b a}\right) - u^0 - \varphi = \theta_X + X(\varphi)$$

$$\Rightarrow\ \log\det\left(g^0_{\bar b a} + \varphi_{\bar b a}\right) = -\overset{\log\det(u^0_{ij})+h}{u^0} - \varphi - \theta_X - X(\varphi)$$

$$\Rightarrow\ \log\det\left(g^0_{\bar b a} + \varphi_{\bar b a}\right) = \log\det\left(g^0_{\bar b a}\right) + h - \varphi - \theta_X - X(\varphi)$$

$$\Rightarrow\ \boxed{\begin{cases} \det\left(g^0_{\bar b a} + \varphi_{\bar b a}\right) = \det\left(g^0_{\bar b a}\right) e^{h - \varphi - \theta_X - X(\varphi)} \\ g^0_{\bar b a} + \varphi_{\bar b a} > 0 \end{cases}}$$

Note: Both θ_X and h are defined wrt g_0!!

Now take everything toric, and let $u = u^0 + \varphi$.
Note that

$$\theta_X + X(\varphi) = X^i \frac{\partial u^0}{\partial x_i} + c_X + X^i \frac{\partial \varphi}{\partial x^i}$$

$$= X^i \frac{\partial u}{\partial x_i} + c_X$$

$$= c_\ell \frac{\partial u}{\partial x_\ell} + c_X.$$

Also have from before

$$\det u^0_{ba} = e^{-h - u^0}$$

So get:

$$\det(u_{ba}) = e^{\overbrace{-u^0 - \varphi}^{-u} - \theta_X - X(\varphi)}$$

Thus in conclusion:

$$\boxed{\det(u_{ba}) = e^{-u - c_\ell \frac{\partial u}{\partial x_\ell} - c_X}}$$

The Estimates

All we need to do is get a uniform estimate for φ in the equation.

$$\begin{cases} \det\left(g^{\circ}_{\bar{k}j} + \varphi_{\bar{k}j}\right) = \det\left(g^{\circ}_{\bar{k}j}\right) e^{h - \theta_X - X(\varphi) - t\varphi} \\ g^{\circ}_{\bar{k}j} + \varphi_{\bar{k}j} > 0 \end{cases}$$

(Wait method of continuity... ~~since~~ so need to get estimate for $(\varepsilon_0, 1] \ni t$!)

In our case, let $u = u^{\circ} + \varphi$. Then if

$$\begin{aligned} W_t &= tu + (1-t)u^{\circ} \\ &= u^{\circ} + t\varphi \end{aligned}$$

and

$$\det\left(g^{\circ}_{\bar{k}j}\right) = e^{-u^{\circ} - h}$$

Thus plugging in, we need

$$\det(u_{ij}) = e^{-u^0 - \phi_x - X(\varphi) - t\varphi}$$

$$\Rightarrow \boxed{\det(u_{ij}) = e^{-w_t - c_\varepsilon \frac{\partial u}{\partial x^2} - cx}} \qquad (*\,*)$$

Thus to get uniform estimate for φ it suffices to get them for w_t $(t \in (\varepsilon_0, 1])$.

<u>Lemma</u>: $\exists C$ ind $t \in [\varepsilon_0, 1]$ s.t.

$$m_t =: \inf_{\mathbb{R}^n} w_t(x) \leq C$$

<u>Proof</u> Define the set A_k by

- $A_k = \{x \in \mathbb{R}^n \mid m_t + k \leq w(x) \leq m_t + k + 1$
- $\bigcup_{i=0}^{k} A_i = \{w \leq m_t + k + 1\}$ is convex, $\forall k \geq 0$

Note: $Dw(\mathbb{R}^n) = \Omega^*$, and $0 \in \Omega$.

A_k is bounded for all k and m_t is attained somewhere in A_0.

The equation (xx) tells us:

$$\det(w_{ij}) = \det(t u_{ij} + (1-t) u^0_{ij})$$
$$\geq t^n \det(u_{ij})$$
$$\geq t^n e^{-cx - d} e^{-w},$$

where $d = \sup\{\sum c_\ell y_\ell \mid y \in \Omega\}$.

Since $t \geq \varepsilon_0$ we have

$$\boxed{\det(w_{ij}) \geq C_0 e^{-m_t} \quad \text{in } A_0,}$$

where $C_0 = \varepsilon_0^n e^{-cx - d - 1}$.

Lemma: $\exists\, T$ affine linear w/ $|T| = 1$ and fixing the center of A_0 such that

$$B_{R/n} \subset T(A_0) \subset B_R$$

Since $|T| = 1$ our equation remains unchanged.

Claim: $R \leq \sqrt{2}\, n\, C_0^{-1/2n} e^{m_t/2n}$

Proof: ~~let~~ Define $v(y)$ by

center of mass of A_0

- $v(y) = \frac{1}{2} C_0^{1/n} e^{-m_t/n} \left[|y - y_t|^2 - \left(\frac{R}{n}\right)^2 \right] + m_t + 1$
- $v_{ij} = C_0^{1/n} e^{-m_t/n} \delta_{ij}$
- $\det(v_{ij}) = C_0 e^{-m_t} \leq \det(w_{ij})$
- On $\partial T(A_0)$ this positive ... so $v \geq w$ on $\partial T(A_0)$

Comparison Principle: $m_t \leq w(y_t) \leq v(y_t)$

Thus :

$$m_t \leq -\frac{1}{2} C_0^{1/n} e^{-m_t/n} \left(\frac{R}{n}\right)^2 + m_t + 1$$

$$\Rightarrow \quad C_0^{1/n} e^{-m_t/n} \left(\frac{R}{n}\right)^2 \leq 2$$

$$\Rightarrow \quad R^2 \leq 2n^2 C_0^{-1/n} e^{m_t/n}$$

$$\Rightarrow \quad \boxed{R \leq \sqrt{2}\, n\, C_0^{-1/2n} e^{m_t/2n}}$$

Now by convexity of w,

$$T(A_k) \subset B_{2(k+1)R}$$ why not $B_{(k+1)R}$?

Thus .

$$\int_{\mathbb{R}^n} e^{-w} = \sum_k \int_{T(A_k)} e^{-w}$$

$$\leq \sum_k e^{-m_t-k} |T(A_k)|$$

$$\leq \omega_n \sum e^{-m_t-k} |2(k+1)R|^n$$

$$= \omega_n \frac{(2R)^n}{e^{m_t}} \sum \frac{(k+1)^n}{e^k} \leq C e^{-m_t}$$

Thus we have

$$e^{-m_t} \geq \frac{1}{C} \int_{\mathbb{R}^n} e^{-w} \, dx$$

But $e^{-w} = e^{cx + c_\ell \frac{\partial u}{\partial x_\ell}} \det(u_{ij})$. Thus

$$e^{-m_t} \geq \frac{1}{C} \int_{\mathbb{R}^n} e^{cx} e^{c_\ell \frac{\partial u}{\partial x_\ell}} \det(u_{ij})$$

$$= \frac{e^{cx}}{C} \int_{\Omega} e^{\sum c_\ell y_\ell} \, dy$$

$$= C' > 0$$

Thus $m_t \leq -\log(C')$. $\blacksquare$

Lemma. Let $x^t = (x_1^t, \dots, x_n^t) \in \mathbb{R}^n$ be the minimal point of $w = w_t$. Then

$$|x^t| \le C$$

for some uniform constant.

Pf: First

$$0 = \int_{\mathbb{R}^n} \frac{\partial w}{\partial x_i} e^{-w} dx = t \int_{\mathbb{R}^n} \frac{\partial u}{\partial x_i} e^{-w} dx + (1-t) \int_{\mathbb{R}^n} \frac{\partial u^0}{\partial x_i} e^{-w} dx$$

But by (**)

$$\int_{\mathbb{R}^n} \frac{\partial u}{\partial x_i} e^{-w} dx = e^{Cx} \int_{\mathbb{R}^n} \frac{\partial u}{\partial x_i} e^{C_\ell \frac{\partial u}{\partial x_\ell}} \det(u_{ij})\, dx$$

$$= e^{Cx} \int_\Omega y_i e^{\Sigma c_\ell y_\ell} dy$$

$$= 0$$

Thus we get

$$\boxed{\int_{\mathbb{R}^n} \frac{\partial u^0}{\partial x_i} e^{-w} dx = 0}$$

Now we want to find a universal C s.t. if $|X_t| > C$, then the last equality ~~breaks~~ breaks

First,

$$\int_{\mathbb{R}^n} e^{-w}\,dx = e^{cx}\int e^{c_i \frac{\partial u}{\partial x_i}} \det(u_{ij})\,dx$$

$$= e^{cx}\int_{\Omega} e^{c_i y_i}\,dy$$

$$= \beta > 0 \quad \underline{\underline{\text{fixed}}}$$

Next, use that $|Dw| \leq d_0 = \sup\{|x| \mid x \in \Omega\}$.

By last lemma we had

$$T(A_0) \subset B_R$$

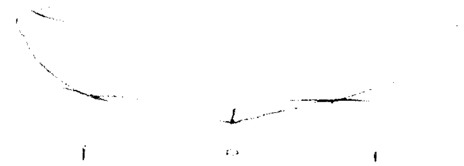
for $R \leq \sqrt{2}\, n C_0^{-1/2n} e^{m_t/2} \leq C$ (ind of t).

I.e.

$$\inf_{\partial B_R(x^t)} w \geq m_t + 1$$

ind of t.

Thus by convexity $|Dw(x)| \geq 1/R$ in $\mathbb{R}^n \setminus B_R(x^t)$.



Thus $e^{-w} \leq e^{-|x-x^t|/R}$ in $\mathbb{R}^n \setminus B_R(x^t)$.

Thus if $\varepsilon > 0$, then exists $R_\varepsilon >> 0$ (and $> R$), such that

const?

$$\int_{\mathbb{R}^n \setminus B_{R_\varepsilon}(x^t)} e^{-w}\,dx \leq \int_{\mathbb{R}^n \setminus B_{R_\varepsilon}(x^t)} e^{-|x-x^t|/R} \leq \varepsilon$$

Now note that u^0 is convex det on $\mathbb{R}^n$ satisfying

$$Du^0 = \Omega, \quad 0 \in \Omega.$$

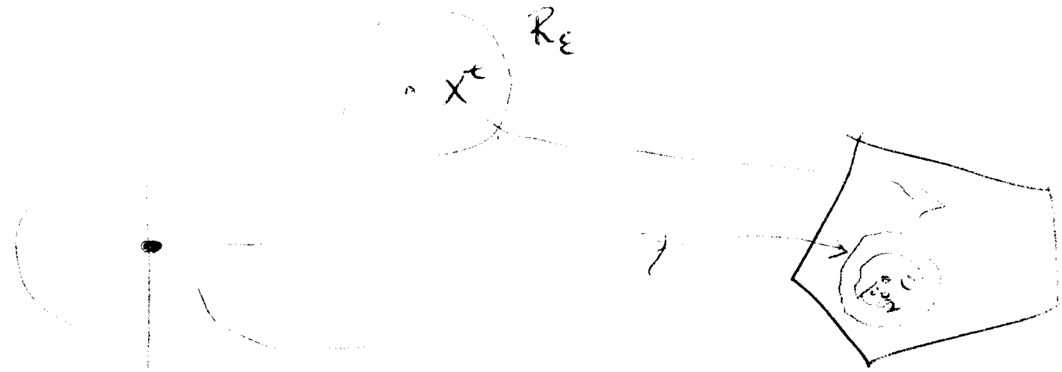
Let $a_0 = \inf d(0, \partial\Omega)$ Now let $\varepsilon > 0$.

If C is chosen so big so that $C - R_\varepsilon$ is mapped beyond $\frac{a_0}{2}$, then $\frac{\partial u^0}{\partial \xi} > \frac{1}{2} a_0$ in $B_{R_\varepsilon}(x^t)$ for $\xi = \frac{x^t}{|x^t|}$.

I.e.

① Fix ε, get R_ε

② Let C be big



Thus we get

$$\int_{B_{R_\varepsilon}(x^t)} \frac{\partial u^c}{\partial \xi} e^{-w} dx \geq \frac{1}{4} \underbrace{a_0}_{\text{approaches } \frac{a_c}{2}} \beta$$

for ε small. (Say $\varepsilon = \frac{\beta}{2}$.)

Also

$$\left| \int_{\mathbb{R}^n \setminus B_{R_\varepsilon}(x^t)} \frac{\partial u^c}{\partial \xi} e^{-w} dx \right| \leq d_0 \varepsilon \quad \text{fixed!}$$

So choose ε really small and get

$$\int_{\mathbb{R}^n} \frac{\partial u^c}{\partial \xi} e^{-w} dx > 0 \quad ↯ \qquad \blacksquare$$

Lemma: Let φ_t solve $(**)$ for $t \in [\varepsilon_0, 1]$.

Then $\sup_M \varphi < C$, C ind $t \in [\varepsilon_0, 1]$.

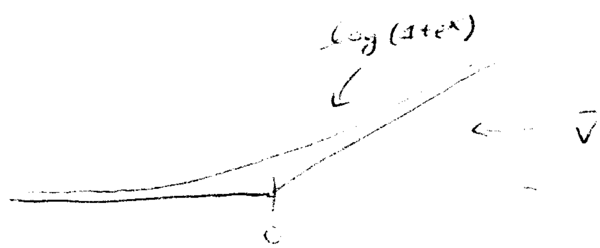
Proof: To help this out, we'll define a function $\bar{v}(x)$ on $\mathbb{R}^n$ by

$$\bar{v}(x) = \max \{ x \cdot p \mid p \text{ vertex of } \Omega \}$$

e.g. If $\mathbb{P}^n$, then

$$\bar{v}(x) = \max \{ 0, x_1, \dots, x_n \}$$

Now note that $|\bar{v} - u^0| < C$. In picture:



So $\bar{v}$ is the „assymptotic cone" of $\bar{v}$.

Now ~~#~~ since $Du(\mathbb{R}^n) = Du^\circ(\mathbb{R}^n) = \Omega$, we have that $\bar{v} + a$ is the assymptotic cone for u, ~~Fix~~ where $a = \cancel{u(0) - u^\circ(0)}$ $u(0) - \bar{v}(0)$

Thus we have

$$\bar{v}(x) + a \geq u(x),$$

and therefore

$$\varphi = u - u^\circ \leq \bar{v} - u^\circ + a \leq C + a.$$

Thus to bound φ, we only need to bound $u(0)$.

Now let x^t be min point of w_t. Then $|x^t| \leq C$. But $|Dw| \leq d_0$ so we get

$$\begin{aligned} w(0) &\leq w(x_t) + C d_0 \\ &\leq C' \end{aligned}$$

But also since from earlier

$$\int_{\mathbb{R}} e^{-w}\,dx = \beta \qquad (\text{top constant})$$

and the fact that $|Dw| \leq d_0$, we get that w cannot be too small at 0 or it must be small in a neighborhood and thus we must have that $w(0) > -C$. This works for every point! Thus $w > -C$ .. So at least get lower bound!

Thus combining we get $|u(0)| < C$ and thus we get what we want. $\blacksquare$

Now all we need ~~d~~ is to show that φ is bdd below.

Lemma: φ is bdd below

Proof: All we need to prove is that

$$\sup\{(\bar{v}-u)(x) \mid x \in \mathbb{R}^n\} \le C.$$

The proof in the last lemma give that $|u(0)| \le C$.

Thus suppose that $u(0) = 0$. This means that u now satisfies:

$$\det(u_{ij}) = \cancel{\exp}\, e^{-Cx - W - \sum C_\ell \frac{\partial u}{\partial x_\ell}} \cdot e^{-Ct}$$

For $r > 0$ denote

$$z(r) = \sup_{|x|=r} (\bar{v}-u)(x).$$

Suppose the supremum is attained at $p = p_r$. Then

$$\begin{aligned} z'(r) &= \lim_{\varepsilon \to 0} \frac{z(r) - z(r-\varepsilon)}{\varepsilon} \\ &\le \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[(\bar{v}-u)(p) - (\bar{v}-u)(p-\varepsilon\xi)\right] \\ &= \partial_\xi (\bar{v}-u)(p), \end{aligned}$$

where $\xi = \frac{p}{|p|}$ is unit vector.