Goals:
- Basic prop. of PSH
- Def. of MA mass
- CLN & Chebyshev
- Comparison Principle

Intrinsic Capacities on Cpt Kähler Mflds

Guedj, Zeriahi

X cpt Kähler

Def: ω a closed, real $(1,1)$ current on X

$Psh(\omega) := \{ \varphi \in L^1(X)$ wrt fixed vol. form, $\varphi : X \to [-\infty, \infty) \mid \omega + i\partial\bar\partial\varphi \geq 0$, φ is ω-u.s.c. $\}$

ω-usc: locally $\omega = i\partial\bar\partial\psi$, $\varphi + \psi$ is usc ($\varphi+\psi$ is psh in $\mathbb{C}^n$-sense)

f is u.s.c. iff $\{x : f(x) < a\}$ is open for all a

Positivity of currents: $(\omega + i\partial\bar\partial\varphi) \wedge i\alpha_1 \wedge \bar\alpha_1 \wedge \dots \wedge i\alpha_{n-1} \wedge \overline{\alpha_{n-1}}$ is a non-neg measure $\forall\, \alpha_1, \dots, \alpha_{n-1}$ (1,0)-forms

Recall:

Def: $u : \Omega \to [-\infty, \infty)$, $\Omega \subset \mathbb{C}^n$, $u \not\equiv -\infty$ is subharmonic if $\forall x_0 \in \Omega$, 1) u is usc. ($\{u<a\}$ open or $u(x_0) \geq \limsup_{x \to x_0} u(x)$) and 2) $u(x_0) \leq$ ~~$\frac{1}{2\pi}\int u(x_0 + re^{i\theta})d\theta$~~ avg. on Balls or spheres whenever

↗ $u(x_0) = \limsup u(x)$

Lemma: u subharmonic $\Rightarrow u \in L^1_{loc}$

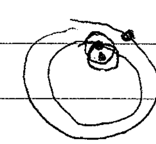
Pf: Let $A = \{x \in U : u$ is L^1 in a nbhd of $x\}$, open by construction

A is ~~open~~ nonempty: $\exists\, x_0$ st $u(x_0) \neq -\infty$, $\overline{B(x_0, r)} \subset U$

u bd'd above on $\bar B$

$\frac{1}{\mathrm{Vol}\, B}\int u \geq u(x_0) > -\infty \Rightarrow x_0 \in A$, A

A is closed: Let $a \in \bar A$, $a' \in A$ st $|a' - a| < \frac{1}{4} d(a, \partial U)$



$\exists\, a''$ close to a' st $u(a'') \neq -\infty$

$u \in L^1(B(a'', r))$, r up to bdry $\Rightarrow a \in A$ □

u subharmonic, $u \in C^2 \Rightarrow \Delta u \geq 0$

more gen, Δu a pos. distribution

Now suppose $u \in L^1_{loc}$, u u-sc.

Note $\Delta u_\varepsilon \geq 0$, so apply previous reasoning to u_ε

$(u_\varepsilon)_\delta(x) \leq (u_\varepsilon)_{\delta'}$ for $\delta \leq \delta'$

Mollifications commute: $(u_\varepsilon)_\delta = (u_\delta)_\varepsilon \Rightarrow (u_\delta)_\varepsilon \leq (u_{\delta'})_\varepsilon$

Let $\varepsilon \to 0 \Rightarrow u_\delta \leq u_{\delta'}$

$\Rightarrow u_\delta \searrow$ as $\delta \to 0$ $\quad \Rightarrow u_\delta \to u$ a.e., L^1 by dom. cvg

$\Rightarrow$ same statement true for $Psh^{(\Omega)}$ fns (since also $i\partial\bar\partial u_\varepsilon \geq 0$)

and therefore also locally for $Psh(X,\omega)$.

For convenience, use $dd^c = \frac{1}{2\pi} i \partial\bar\partial$ from now on

Basic properties of $Psh(X,\omega)$:

Prop: 1) $\omega_1 \leq \omega_2 \Rightarrow Psh(X,\omega_1) \subseteq Psh(X,\omega_2)$

2) $\forall A \in \mathbb{R}_{>0}$, $Psh(X, A\omega) = A\,Psh(X,\omega)$

3) $\omega' = \omega + dd^c\chi \Rightarrow Psh(X,\omega') = Psh(X,\omega) + \chi$

4) $\varphi, \psi \in Psh(X,\omega)$, then $\max(\varphi,\psi)$, $\frac{\varphi+\psi}{2}$, $\log(e^\varphi + e^\psi) \in Psh(X,\omega)$

Rmks on Pf: 1), 2), 3) immediate

$$4) \Rightarrow dd^c \log(e^\varphi + e^\psi) = \frac{e^\varphi dd^c\varphi + e^\psi dd^c\psi}{e^\varphi + e^\psi} + \frac{e^{\varphi+\psi} d(\varphi-\psi)\wedge d^c(\varphi-\psi)}{(e^\varphi+e^\psi)^2}$$

& $df \wedge d^c f \geq 0$, $\quad \max(\varphi,\psi) = \lim \frac{1}{j}\log(e^{j\varphi} + e^{j\psi})$ $\quad \square$

Essentially, $Psh(X,\omega)$ just dep. on cohomology class of ω.

Prop: Let $T_{[\omega]}(X)$ denote the set of positive closed $(1,1)$ currents ω' cohomologous to ω

Then $Psh(X,\omega) \simeq T_{[\omega]}(X) \oplus \mathbb{R}$

Monge-Ampère Capacity

Now we want $[\omega]$ pos, so let ω Kähler

Want to define for $\varphi \in Psh(X,\omega)$ a MA mass like $\omega_\varphi^n = (\omega + dd^c\varphi)^n$

In general, wedge products of currents not defined

Let T pos closed (p,p)-current (diff. form w/ measure coeff.)

total variation $\|T\| := \int_X T \wedge \omega^{n-p}$

~~Want~~ For $\varphi \in Psh(X,\omega)$, write $\varphi \in L^1(T)$ if φ is integrable wrt each coeff. of T

$\iff \varphi$ int. wrt $T \wedge \omega^{n-p}$

Define $dd^c\varphi \wedge T := dd^c(\varphi T)$ for $\varphi \in Psh(X,\omega) \cap \underline{\underline{L^\infty}}$

So $\omega_\varphi \wedge T = \omega \wedge T + dd^c(\varphi T)$

Prop: ~~$dd^c\varphi$~~ $\omega_\varphi \wedge T$ is a closed, pos. current.

Pf: Locally, let $\varphi_{1/k} = \varphi * \chi_{1/k}$ be seq. of smooth psh fns converging to φ. $\varphi \leq \varphi_{1/k} \leq \varphi_1$, so by dom. cvg, $\varphi_{1/k} T$ conv. weakly to φT

$\Rightarrow$ also $dd^c(\varphi_{1/k} T)$ conv. weakly to $dd^c(\varphi T)$

Since $\varphi_{1/k}$ smooth, $dd^c(\varphi_{1/k} T) = dd^c\varphi_{1/k} \wedge T$ in usual sense

Locally $\omega_{\varphi_{1/k}} \wedge T = dd^c(\psi + \varphi_{1/k}) \wedge T$

$T \geq 0$ and $dd^c(\psi + \varphi_{1/k})$ a pos. $(1,1)$-form

$\Rightarrow \omega_{\varphi_{1/k}} \wedge T \geq 0$

Hence the weak limit $\omega_\varphi \wedge T \geq 0$ and closed □

Can now define inductively $\omega_\varphi^j \wedge T$, let $j = n$, $T = 0$ to get MA operator ω_φ^n

Prop: $\Psi \in Psh(X,\omega)$, $\sup \Psi \le 0$.

Then $Cap_\omega(\{\Psi < -t\}) \le \frac{1}{t}\left[\int_X (-\Psi)\omega^n + n\,Vol_\omega(X)\right] \quad \forall t > 0$.

In particular, if P is a $Psh(X,\omega)$-polar set then $Cap_\omega(P) = 0$

Pf. Fix $\varphi \in Psh(X,\omega)$ st $0 \le \varphi \le 1$. Fix $t>0$, set $K_t = \{\Psi < -t\}$

$$\int_{K_t} \omega_\varphi^n \le \int_X (-\Psi/t)\,\omega_\varphi^n \overset{\text{Chebyshev}}{\le} \frac{1}{t}\left[\int_X (-\Psi)\omega^n + n\,Vol_\omega(X)\right] \quad \text{by previous cor.}$$

Now take sup over all φ's

Now want to prove Comparison Principle: $\varphi, \Psi \in Psh(X,\omega) \cap L^\infty(X)$ (Bedford-Taylor)

$$\Rightarrow \int_{\{\varphi<\Psi\}} \omega_\Psi^n \le \int_{\{\varphi<\Psi\}} \omega_\varphi^n$$

Pf: First, smooth case. Assume $\{\varphi<\Psi\}$ in some coord nbhd

$$\omega_\Psi^n - \omega_\varphi^n = (\omega_\Psi - \omega_\varphi)\wedge\left(\sum \omega_\Psi^i \wedge \omega_\varphi^{n-1-i}\right)$$
$$= dd^c(\Psi-\varphi)\wedge\Theta, \quad \Theta \text{ a pos. closed } (n-1,n-1)\text{-form}$$

$$\Rightarrow \int_{\{\varphi<\Psi\}} \omega_\Psi^n - \omega_\varphi^n = \int_{\partial\{\varphi<\Psi\}} d^c(\Psi-\varphi)\wedge\Theta \,; \qquad ~~Must~~ \text{ show } d^c(\Psi-\varphi)\wedge\Theta = f\,d\sigma,\ f\le 0.$$

Let $\rho \in C^\infty$ be a defining function for $\{\varphi<\Psi\}$, ie $\{\varphi<\Psi\} = \{\rho<0\}$, $\partial\{\varphi<\Psi\} = \{\rho = 0\}$ and $\nabla\rho \ne 0$ on $\partial\{\varphi<\Psi\}$. Since $\varphi = \Psi$ on bdry, $\Psi - \varphi = \alpha\rho$, $\alpha \le 0$ in $\overline{\{\varphi<\Psi\}}$.

~~$\Rightarrow (d^c\alpha = 0)$?~~

$$d^c(\Psi-\varphi) = d^c(\alpha\rho) \overset{\alpha d^c\rho + \rho d^c\alpha,\ \rho=0 \text{ on } \partial\{\varphi<\Psi\}}{=} \alpha d^c\rho \Rightarrow \int_{\{\varphi<\Psi\}} \omega_\Psi^n - \omega_\varphi^n = \int_{\partial\{\varphi<\Psi\}} \alpha\, d^c\rho\wedge\Theta$$

σ the surface area form $\Rightarrow \sigma = \frac{*d\rho}{\|d\rho\|}$, $*$ the Hodge star in $\mathbb{C}^n$

$$\alpha\, d^c\rho\wedge\Theta = f\,d\sigma \Rightarrow \alpha\, d\rho\wedge d^c\rho\wedge\Theta = f\, d\rho\wedge\frac{*d\rho}{\|d\rho\|}, \quad d\rho\wedge d^c\rho \text{ pos.} \Rightarrow \text{if } \alpha\le 0, \text{ so is } f.$$

Cor 3.8 (Quasicontinuity) Let $\varphi \in Psh(X,\omega)$. $\forall \varepsilon > 0$ $\exists$ an open set O_ε of X st $Cap_\omega(O_\varepsilon) < \varepsilon$ and φ is continuous on $X \setminus O_\varepsilon$.

Pf: For $t > 0$ suff. large, $O_1 = \{\varphi < -t\}$ has capacity $< \varepsilon/2$ by Prop 3.6.
In $X \setminus O_1$, can replace φ by $\varphi_t = \max(\varphi, -t)$, bounded

Błocki-Kol (~~Demailly~~) HARD

• Global regularization: $\exists$ seq. ψ_j of smooth $Psh(X, A\omega)$ decreasing to φ_t on X for some $A \geq 1$

By above prop., $O_j = \{\psi_{k_j} > \varphi_t + \frac{1}{j}\}$ has capacity $< \varepsilon 2^{-j-1}$ for k_j suff. large

Now ψ_{k_j} conv. unif to $\varphi = \varphi_t$ on $X \setminus O_\varepsilon$, $O_\varepsilon = \bigcup_{j=1}^{\infty} O_j$
so φ is continuous on $X \setminus O_\varepsilon$, $Cap(O_\varepsilon) \leq \varepsilon$.

Ex: $Cap_\omega(\cdot)$ does not distinguish b/w big sets
Assume $\exists$ ~~Let~~ D an ample div. st $[D] \sim k\omega$, $k \in \mathbb{N}$. Then $\exists$ $\varphi \in Psh(X,\omega)$
st $dd^c \varphi = k^{-1}[D] - \omega$
$\Rightarrow$ $\varphi \in C^\infty(X \setminus D)$, $e^\varphi \in C^0(X)$, and $\{\varphi = -\infty\} = D$
May assume $\sup_X \varphi = 0$.
Let $\varphi_c = \max(\varphi, -c) \in Psh(X,\omega) \cap C^0(X)$. Then $\varphi_c = \varphi$ outside $V_c = \{\varphi < -c\}$, a nbhd of D
$0 \leq 1 + \varphi_1 \leq 1$ and $\omega_{1+\varphi_1} = \omega_\varphi = 0$ in $X \setminus V_1$

$$\Rightarrow Cap_\omega(X) = \int_X (\omega_{1+\varphi_1})^n = \int_{V_1} (\omega_{1+\varphi_1})^n \leq Cap_\omega(V_1) \Rightarrow Cap_\omega(V_1) = Cap_\omega(X)$$

$X = \mathbb{P}^n$, $\omega = \omega_{FS}$, $D = H_\infty$, $\varphi([z:t]) = \log|t| - \frac{1}{2}\log(\|z\|^2 + |t|^2)$
$\mathbb{P}^n \setminus V_1 = \{z \in \mathbb{C}^n \mid |z| \leq \sqrt{e^2 - 1}\}$

Thus $\int_{\{\varphi \le \psi\}} \omega_\varphi^n \ge \lim_{k\to\infty} \int_{\{\varphi \le \psi\}\cap F} \omega_{\varphi_k}^n$

$$\ge \lim_{k\to\infty} \int_{\{\varphi_k \le \psi\}} \omega_{\varphi_k}^n - \int_G \omega_{\varphi_k}^n$$

$\therefore$ if $M \ge 2\sup\{|\varphi_k(z)|\}$

Then $\int_{\{\varphi < \psi\}} \omega_\varphi^n \le \int_{\{\varphi \le \psi\}} \omega_\varphi^n + 3M^n\varepsilon$

$\varepsilon > 0$ arbitrary $\Rightarrow \int_{\{\varphi < \psi\}} \omega_\varphi^n \le \int_{\{\varphi \le \psi\}} \omega_\varphi^n$

Replace φ by $\varphi + \eta$, η const. $\{\varphi + \eta < \psi\} \nearrow \{\varphi < \psi\}$ as $\eta \downarrow 0$

and $\{\varphi + \eta \le \psi\} \nearrow \{\varphi < \psi\}$

and $\omega_{\varphi+\eta}^n = \omega_\varphi^n$

$\square$

Prop 2.6: Assume ω smooth. $\varphi_j \in Psh(X,\omega)$

(1) If (φ_j) is unif. bd'd from above on X, then either φ_j converges unif. to $-\infty$ on X, or (φ_j) is relatively cpt in $L^1(X)$

(2) If $\varphi_j \to \varphi$ in $L^1(X)$, then φ coincides a.e. w/ a unique fn $\varphi^* \in Psh(X,\omega)$

$$\sup \varphi^* = \lim_{j\to\infty} \sup \varphi_j$$

(3) In particular, if φ_j is decreasing, then either $\varphi_j \to -\infty$ or $\varphi = \lim \varphi_j \in Psh(X,\omega)$

Sim, if φ_j is increasing and unif. bd'd from above, then $\varphi := (\lim \varphi_j)^* \in Psh(X,\omega)$

$$u^* = \lim_{\varepsilon\to 0} \sup_{B(z,\varepsilon)} u$$

Prop 2.7: Assume ω smooth.

Then $\tilde{\mathcal{F}}_0 := \{\varphi \in Psh(X,\omega) \mid \sup \varphi = 0\}$ is a cpt subset of $Psh(X,\omega)$

→ Prop 2.1 in Dinew

$\omega < \Omega$. Write

$$dd^c u_j + \omega = dd^c u_j + \Omega - T, \quad T = (\Omega - \omega).$$

Then by the Newton expansion

$$(2.2) \qquad (dd^c u_j + \omega)^n = (dd^c u_j + \Omega)^n - n(dd^c u_j + \Omega)^{n-1} \wedge T + \ldots \pm T^n.$$

By the convergence theorem for psh functions [4] all the terms on the right converge as currents, and the sum of their limits is

$$(dd^c u + \Omega)^n - n(dd^c u + \Omega)^{n-1} \wedge T + \ldots \pm T^n = (dd^c u + \omega)^n.$$

We note that *all* functions u in $PSH(X, \omega)$, normalized by the condition $sup_X u = 0$ are uniformly integrable. This follows from classical results in potential theory and psh-like property as in [22]. Since such results seem to be important in a more general setting (compare [29]) we give here a complete argument following quite closely the one in [18], where the authors treat the Kähler case.

Proposition 2.1. *Let $u \in PSH(X, \omega)$ be a function satisfying $sup_X u = 0$. Then there exists a constant C dependent only on X, ω such that*

$$\int_X |u| \, \omega^n \leq C.$$

Proof. Consider a double covering of X by coordinate balls $B_s^1 \subset\subset B_s^2 \subset X$, $s = 1, \cdots, N$. In each B_s^2 there exists a strictly plurisubharmonic potential ρ_s satisfying the following properties:

$$(2.3) \qquad \begin{cases} \rho_s|_{\partial B_s^2} = 0 \\ inf_{B_s^2} \rho_s \geq -C_1 \\ dd^c \rho_s = \omega_{2,s} \geq \omega, \end{cases}$$

where C_1 is a constant dependent only on the covering and ω. Suppose now that there exist a sequence $u_j \in PSH(X, \omega), sup_X u_j = 0$ satisfying $lim_{j \to \infty} \int_X |u_j| \, \omega^n = \infty$. After choosing subsequence (which for the sake of brevity we still denote by u_j) we may assume that

$$(2.4) \qquad \int_X |u_j| \, \omega^n \geq 2^j$$

and moreover a sequence of points x_j where u_j attains maximum is contained in some fixed ball B_s^1.

Note that $\rho_s + u_j$ is an ordinary plurisubharmonic function in B_s^2 and by the sub mean value property one has

$$(2.5) \qquad \rho_s(x_j) = \rho_s(x_j) + u_j(x_j) \leq C_2 \int_{B_s^2} \rho_s(z) + u_j(z) d\lambda \leq C_2 \int_{B_s^2} u_j(z) d\lambda + C_3,$$

where $d\lambda$ is the Lebesgue measure in the local coordinate chart, while C_2, C_3 are constants dependent only on B_s^1 and B_s^2. Thus (2.5) implies that for some constant C_4 one has

$$(2.6) \qquad \int_{B_s^2} |u_j(z)| d\lambda \leq C_4.$$

Consider the function $v := \sum_{j=1}^{\infty} \frac{u_j}{2^j}$. By classical potential theory this is again a ω-psh function or constantly $-\infty$. By (2.6) however the integral of v over B_s^2 is finite thus it is a true ω-psh function. Reasoning like in the fixed ball B_s^2 we easily obtain that $v \in L^1(B_t^1)$ for any $t \in 1, \cdots, N$ and hence $v \in L^1(X)$. This contradicts (2.4), and thus the existence of an uniform bound is established. □

Demailly **Prop:** The convex cone $Sh(\Omega) \cap L^1_{loc}(\Omega)$ is closed in $L^1_{loc}(\Omega)$ and every bounded subset is relatively compact.

Pf. (Note: we are concerned w/ cpt mflds X, $Psh(X,\omega) \subset L^1(X)$)

Suppose $\|u_j\|_{L^1(X)}$ unif. bd'd $\Rightarrow \|u_j\|_{L^1(K)}$ unif. bd'd $\forall$ cpt subsets

Let $\mu_j = \Delta u_j \geq 0$. if $\psi \in C^\infty(\Omega)$ equal to 1 in a nbhd ω of K, $0 \leq \psi \leq 1$

$$\mu_j(K) \leq \int_\Omega \psi \Delta u_j \, d\lambda = \int_\Omega \Delta\psi \, u_j \, d\lambda \leq C \|u_j\|_{L^1(K')} \quad \text{where } K' = \operatorname{supp} \psi.$$

$\Rightarrow \mu_j$ unif. bd'd in mass on every cpt subset of Ω

$\Rightarrow$ by weak compactness, (μ_{j_ν}) cvgs weakly to pos. meas μ on Ω

Claim: $f * (\psi\mu_{j_\nu}) \to f * (\psi\mu)$ in $L^1_{loc}(\mathbb{R}^m)$ $\forall f \in L^1_{loc}(\mathbb{R}^m)$

Clear for $f \in C^\infty$

Gen, use C^∞ approx, and $\|(f-g) * (\psi\mu_{j_\nu})\|_{L^1(A)} = \int_A \left| \int_\Omega (f-g)(y) \, \Delta\psi(x-y) u_j(x-y) \, dy \right| dx$ ($z = x-y$)

$$\leq \int_A \int_{K'} |(f-g)|(z+x) \, |\Delta\psi \, u_j(z)| \, dz \, dx = \int_{A+K'} |f-g| \cdot \int_{K'} \Delta\psi \, u_j \leq \|f-g\|_{L^1(A+K')} \mu_{j_\nu}(K')$$

$x-y \in K'$

$x \in A$

$-y \in K' - A$

$y \in A - K'$

Now apply for $f = N$, the Newton Kernel ($N \in L^1_{loc}$, $\Delta N * g = g$)

Let $h_j = u_j - N * (\psi\mu_j)$

$\Delta h_j = \Delta u_j - \psi\mu_j = 0$ in ω $\Rightarrow h_j$ harmonic on ω and bd'd in $L^1(\omega)$

$h_j = h_j * \rho_\varepsilon$ for any smoothing kernel $\Rightarrow D^\alpha h_j = h_j * (D^\alpha \rho_\varepsilon)$

so all derivatives unif. bdd in ω

$\Rightarrow$ after extracting new subseq, h_{j_ν} converges unif. to limit h on ω

$\Rightarrow u_{j_\nu} = h_{j_\nu} + N * (\psi\mu_{j_\nu})$ converges to $u = h + N * (\psi\mu)$ in $L^1_{loc}(\omega)$.

X cpt, ω Kähler

Siciak Extremal Fn, Alexander-Taylor's Thm

Def: K a Borel subset of X

Set $V_{K,\omega} := \sup\{\varphi(x) \mid \varphi \in Psh(X,\omega),\ \varphi \le 0 \text{ on } K\}$ (def. here w/o usc reg.)

Thm: K as above (u.s.c reg = $\lim_{\varepsilon\to 0} \sup_{B(x,\varepsilon)} V_{K,\omega}$)

(1) K is Psh(X,ω)-polar iff $\sup_X V^*_{K,\omega} = +\infty$ iff $V_K^* \equiv +\infty$

(2) If K is not Psh(X,ω)-polar, then $V_K^* \in Psh(X,\omega)$, $V_K^* \equiv 0$ in interior of K,

$(\omega_{V_K^*})^n = 0$ in $X \setminus \bar{K}$

and $\int_{\bar{K}} (\omega_{V_K^*})^n = \int_X \omega^n = Vol_\omega(X)$

Pf: (1) Assume $\sup_X V_K^* = +\infty$. ∃ an increasing seq. $\varphi_j \in Psh(X,\omega)$ st $\varphi_j = 0$ on K and $V_K^* = (\lim \nearrow \varphi_j)^*$ (Lemma of Choquet)

May assume $\sup \varphi_j \ge 2^j$

Set $\psi_j = \varphi_j - \sup\varphi_j \Rightarrow \{\psi_j\} \subset \mathcal{F}_0$ compact

μ a smooth vol. form $\Rightarrow \int \psi_j \, d\mu \ge -C_\mu \ \forall j$. Set $\Psi := \sum_{j\ge1} 2^{-j}\psi_j$

Then $\Psi \in Psh(X,\omega)$ as a decreasing limit of fns in PSH w/ $\int_X \Psi d\mu \ge -C_\mu > -\infty$

For $x \in K$, $\Psi(x) = -\sum_{j\ge1} 2^{-j}\sup\varphi_j = -\infty \Rightarrow K \subset \{\Psi = -\infty\}$

If K is PSH(X,ω)-polar, $K \subset \{\Psi = -\infty\}$ for some $\Psi \in Psh(X,\omega)$

Then for all $c \in \mathbb{R}$, $\Psi + c \in Psh(X,\omega)$ and $\Psi + c \le 0$ on K

$\therefore V_{K,\omega} \ge \Psi + c \quad \forall c \in \mathbb{R}$

$\Rightarrow V_K = +\infty$ on $X \setminus \{\Psi = -\infty\}$, hence $V_K^* \equiv +\infty$ on X since $\{\Psi = -\infty\}$ has zero volume

(2) Suppose K not Psh(X,ω)-polar. Then $V^*_{K,\omega} \in Psh(X,\omega)$ & $V_K^* = 0$ in K°.

Let $\varphi_j \in Psh(X,\omega)$ ~~st~~ $V^*_{K,\omega} = (\lim \nearrow \varphi_j)^*$, $\varphi_j = 0$ on K. Fix B a small ball in $X \setminus \bar{K}$. Let $\hat{\varphi}_j$ be a soln of the Dirichlet problem w/ bdry values φ_j (Bedford-Taylor)

$\rightarrow \hat{\varphi}_j \in Psh(X,\omega)$, $\hat{\varphi}_j = \varphi_j$ on $X \setminus B \Rightarrow \hat{\varphi}_j = 0$ on K $\Rightarrow \hat{\varphi}_j \le V_{K,\omega}$, $\hat{\varphi}_j$ increasing

$\rightarrow (\omega_{\hat{\varphi}_j})^n = 0$ in B and $\lim \nearrow \hat{\varphi}_j = V_K^* \Rightarrow (\omega_{V_K^*})^n = 0$ in B

(MA cts on inc. seq.)

Alexander Capacity

Def. K a Borel subset of X. Set $T_\omega(K) := \exp(-\sup_X V^*_{K,\omega})$

Prop: P a Borel subset. Then $T_\omega(P) = 0$ iff P is $Psh(X,\omega)$-polar.

Moreover, if $\varphi \in Psh(X,\omega)$, then $T_\omega(\varphi < -t) \leq C_\varphi \exp(-t)$ $\forall t \in \mathbb{R}$

where $C_\varphi = \exp(-\sup \varphi)$

Pf. Let $\varphi \in Psh(X,\omega)$, let $K_t = \{\varphi < -t\}$. Then $\varphi + t \leq 0$ on $K_t \Rightarrow \varphi + t \leq V^*_{K_t,\omega}$

$\Rightarrow \sup \varphi + t \leq \sup V^*_{K_t}$

$$\Rightarrow T_\omega(K_t) = \exp(-\sup_X V^*_{K_t,\omega})$$
$$\leq \exp(-\sup\varphi - t)$$

First assertion follows from (1) of prev. Thm.

Now assume ω Kähler, $Vol_\omega(X) = 1$.

Prop: $\exists A > 0$ st $\forall K$ cpt $\subset X$, $\exp\left(\frac{-A}{Cap_\omega(K)}\right) \leq T_\omega(K) \leq e \cdot \exp\left(\frac{-1}{Cap_\omega(K)^{1/n}}\right)$

Pf. Set $M_K = \sup V_{K,\omega}$. If $M_K = +\infty$, then K is $Psh(X,\omega)$-polar, nothing to prove, since $T_\omega(K) = Cap_\omega(K) = 0$.

So assume $M_K < \infty \Rightarrow V^*_{K,\omega} \in Psh(X,\omega)$.

$\omega + dd^c(M_K u_K) = \omega + M_K dd^c u_K \leq M_K(\omega + dd^c u_K)$

If $M_K \geq 1$, set $u_K = \frac{1}{M_K} V^*_{K,\omega}$, $0 \leq u_K \leq 1$. $\omega_{V^*_{K,\omega}} \leq M_K \omega_{u_K}$

$$\Rightarrow \frac{1}{M_K^n} = \frac{1}{M_K^n}\int_K (\omega_{V^*_{K,\omega}})^n \leq \int_K (\omega_{u_K})^n \leq Cap_\omega(K)$$

$$\Rightarrow T_\omega(K) = \exp(-M_K) \Rightarrow T_\omega(K) \leq \exp\left(\frac{-1}{Cap_\omega(K)^{1/n}}\right)$$

If $0 \leq M_K \leq 1$, then $0 \leq V^*_{K,\omega} \leq 1$

$$\Rightarrow 1 = \int_K (\omega_{V^*_{K,\omega}})^n \leq Cap_\omega(K) \leq Cap_\omega(X) = 1, \quad T_\omega(K) \leq T_\omega(X) = 1$$
$$\Rightarrow T_\omega(K) \leq e \cdot \exp(-1).$$

Note: Valentino's pf of L^∞ est. only used this ineq., not $T_\omega(K) \geq \exp\left(\frac{-A}{Cap_\omega(K)}\right)$.

~~[illegible]~~