Bando & Mabuchi, Uniqueness of Einstein Kähler Metrics Modulo Connected Group Actions

X cpt Kähler w/ $c_1(X) > 0$.

Let $\mathcal{K} = \{\omega \mid [\omega] = 2\pi c_1(X)\}$ Kähler forms

$\mathcal{K}^+ := \{\omega \in \mathcal{K} \mid \omega$ has pos. def. Ricci tensor$\}$

$\mathcal{E} = \{\omega \in \mathcal{K} \mid \omega$ is KE$\}$

$G = \mathrm{Aut}^0(X)$, component of identity

~~Thm A: Fix $\omega_1 \in \mathcal{K}$~~

$V = n! \int \omega_0^n/n! = n! \cdot V_0$

Also recall ~~the~~ functionals: Mabuchi energy $M(\varphi_1, \varphi_2) = -\int_a^b \int_X \dot{\varphi}_t (R - n)\, \omega_{\varphi_t}^n / V \, dt$ (R: scalar curvature)

taken along any path of potentials from φ_1 to φ_2

Thm A: Fix $\omega_1 \in \mathcal{K}$. Let $\mu^+ : \mathcal{K}^+ \to \mathbb{R}$ be the restriction to $\mathcal{K}^+$ of the Mabuchi $(K\text{-})$ energy map $\omega \mapsto M(\omega_1, \omega)$.

Assume $\mathcal{E} \neq \emptyset$.

Then (i) μ^+ is bd'd from below and takes its absolute min on $\mathcal{E}$

(ii) $\mathcal{E}$ consists of a single G-orbit.

$$\frac{\det . g_{i\bar j} + \varphi_{i\bar j}}{\det g_{i\bar j}} = e^{-\varphi + tF}$$

at max, $e^{-\varphi+tF} \le 1 \Rightarrow -\varphi + tF \le 0$

$\Rightarrow \sup \varphi \ge tF \cancel{\sup tF} \ge \inf tF$

at min $e^{-\varphi+tF} \ge 1 \Rightarrow -\varphi + tF \ge 0$

$\inf \varphi \le tF \le \sup tF$

Recall eqn for KE metric:

$$R_{i\bar j} = -\partial_i \partial_{\bar j} \log \det g$$

$$R_{i\bar j} - c\, g_{i\bar j} = \partial_i \partial_{\bar j} F$$

$$R'_{i\bar j} = c\, g'_{i\bar j}, \quad g'_{i\bar j} = g_{i\bar j} + \partial_i \partial_{\bar j} \varphi$$

$$\Rightarrow R'_{i\bar j} - R_{i\bar j} = \cancel{\log \det (g + \partial_{i\bar j}) =}$$

$$= -\partial_i \partial_{\bar j} (\log \det g' - \log \det g)$$

$$= c\, g'_{i\bar j} - (\partial_i \partial_{\bar j} F + c\, g_{i\bar j})$$

$$= c(\partial_i \partial_{\bar j} \varphi) - \partial_i \partial_{\bar j} F$$

$$\Rightarrow \cancel{\log \det (g+} \log \frac{\det g + \varphi_{i\bar j}}{\det g} = -c\varphi + F + \cancel{A}$$

Solve by method of continuity in $c=0$, $c=-1$ case: $\frac{\det g + \varphi_{i\bar j}}{\det g} = \underset{\text{only in } c=0 \text{ case}}{A_t}\, e^{-c\varphi + tF}$

Problem in $c=1$ case: No C^0 estimate in general

$\boxed{\mu(\omega) := M(\omega_0, \omega)}$

Notation of Bando-Mabuchi

$R(\omega)$ = Ricci form of ω

$\omega_0(\varphi)$ = form corr. to $(g_0)_{i\bar j} + \partial_i \partial_{\bar j} \varphi$

also $\sigma(\omega) = R$, scalar curvature

$\Omega_0(\varphi) := e^{-\varphi} \tilde\omega^n$

where $\tilde\omega \in \mathcal{K}^+$ unique st $R(\tilde\omega) = \omega_0$

<u>Idea</u>: Consider two eqns for using method of continuity.

(1) $\log \dfrac{\det(g_{i\bar j} + (\varphi_t)_{i\bar j})}{\det g_{i\bar j}} = -t\varphi_t + F \qquad t \in [0,1]$

(2) $\log \dfrac{\det(g_{i\bar j} + (\varphi_t)_{i\bar j})}{\det g_{i\bar j}} = -t\varphi_t - L(0, \varphi_t) + F \quad t \in [0,1]$, $\quad L(\varphi', \varphi'') = \int_a^b \int_X \dot\varphi_s\, \omega_{\varphi_s}^n / V\, ds$

generalized Aubin equation

Will find for each orbit, one ~~such~~ $\theta \in \mathcal{O}$ st can extend backwards to something given $\Rightarrow$ uniqueness of orbit
indep. of orbit

If $\exists\, \theta$ a KE metric, then (1) has a solution ψ_1 at $t=1$,

ie $\theta = \omega_0 + i\partial\bar\partial\psi_1$, ψ_1 may fail to extend to family ψ_t for $t \in [1-\varepsilon, 1]$

~~If such a family exists,~~

~~Differentiate (1) wrt t at $t=1$: $\boxed{(\Delta_\theta + 1)\dot\psi_{t|t=1} = -\psi_1}$~~

~~$\Rightarrow \int_X \psi_1 \varphi\, \theta^n = 0 \quad \forall\, \varphi \in H_\theta$, where $H_\theta = \mathrm{Ker}(\Delta_\theta + 1) \subset C^\infty(X)_{\mathbb{R}}$~~

~~$A(\varphi) := \log(\omega_0(\varphi)^n/\omega_0^n)$~~ $A(\varphi) := \log(\omega_0(\varphi)^n/\omega_0^n)$

~~(4.1.1) $A(\varphi_t) = -t\varphi_t - L(0$~~

(1) $A(\psi_t) = -t\psi_t + f$

Still

Considering Eqns (1) & (2)

(2) $A(\varphi_t) = -t\varphi_t - L(0,\varphi_t) + f$

$$(\mathrm{Ric}_{\varphi_t})_{i\bar j} = -\partial_i\partial_{\bar j} A(\psi_t) - \partial_i\partial_{\bar j}\log\det g_0$$
$$= +t\,\partial_i\partial_{\bar j}\psi_t - \partial_i\partial_{\bar j} f - \partial_i\partial_{\bar j}\log\det g_0$$
$$= (\mathrm{Ric}_0)_{i\bar j} - \partial_i\partial_{\bar j} f + t\,\partial_i\partial_{\bar j}\psi_t$$
$$= g_{i\bar j} + t\,\partial_i\partial_{\bar j}\psi_t = (1-t)g_0 + tg$$
$$=$$

For $t \in (0,1]$, let $j'(\varphi_t) := \varphi_t + t^{-1}L(0,\varphi_t)$, $\quad j''(\psi_t) := \psi_t - (t+1)^{-1}L(0,\psi_t)$

(this can be used for $t \in [0,1]$)

$A(j'(\varphi_t)) = A(\varphi_t)$, etc.

eqn $A(j'(\varphi_t)) \overset{A(\varphi_t)}{=} -t\varphi_t - L(0,\varphi_t) + f \overset{A(\varphi_t)}{=} -t(j'(\varphi_t)) + f \Rightarrow j'(\varphi_t)$ solves (1)

Check: $\underline{j' \circ j'' = \mathrm{id}}$:

$$j'(\psi_t - (t+1)^{-1}L(0,\psi_t)) = \psi_t - \tfrac{1}{t+1}L(0,\psi_t) + \tfrac{1}{t}L\left(0, \psi_t - \tfrac{1}{t+1}L(0,\psi_t)\right)$$
$$= \psi_t - \tfrac{1}{t+1}L(0,\psi_t) + \tfrac{1}{t}\int_0^1\!\!\int_X \left(\psi_t - \tfrac{1}{t+1}L(0,\psi_t)\right)\omega^n_{s\psi_t}/V\, ds$$
$$\omega^n_{s\psi_t} = (\omega_0 + s\,i\partial\bar\partial\psi_t)^n$$
$$= \quad '' \quad + \tfrac{1}{t}\left(L(0,\psi_t) - \tfrac{1}{t+1}L(0,\psi_t)\right)$$
$$= \psi_t - \tfrac{t}{t(t+1)}L(0,\psi_t) + \left(\tfrac{t+1}{t(t+1)} - \tfrac{1}{t(t+1)}\right)L(0,\psi_t)$$
$$= \psi_t \quad \checkmark$$

$\boxed{\therefore \text{ For } t \neq 0, \text{ can pass b/w solutions of (1) \& (2)}}$

Also, for both eqns (1) & (2), $(\mathrm{Ric}_t)_{i\bar j} = (1-t)(g_0)_{i\bar j} + t(g_t)_{i\bar j}$

$\Rightarrow$ Ricci is pos.-def. for all t

For $t=0$,
Yau's theorem $(c=0) \Rightarrow$ Eqn (1) has a unique solution.

Cor: For $t=0$, Eqn (2) has a unique solution φ_0.
Moreover $L(0,\varphi_0)=0$ and $\cancel{(Ric_{\omega(\varphi_0)})_{i\bar{j}} = (g_0)_{i\bar{j}}}$ (not KE!) since $\varphi_0 \neq 0$
$Ric(\omega_0(\varphi_0)) = \omega_0 \neq \omega_0(\varphi_0)$

Pf: Yau's thm gives existence of sol. ψ_0 to eqn 1.
then $A(\psi_0) = A(j''(\psi_0))$, $\cancel{t} j''(\psi$.
f

$$-L(0, j''(\psi_0)) = -L(0, \psi_0 - L(0,\psi_0)) = 0 \Rightarrow j''(\psi_0) \text{ solves eqn (2) at } t=0.$$

For uniqueness, let φ_0 be a solution at $t=0$.

$$\Rightarrow \log\frac{\det g'}{\det g_0} = -L(0,\varphi_0)+f$$

$$\Rightarrow \int \omega_0^n = \int \det g_0 = \int \frac{\det g'}{\det g_0}\det g_0 = e^{-L(0,\varphi_0)}\int e^f \det g_0$$

$$\int e^f \det g_0 = \int \det g_0 \Rightarrow L(0,\varphi_0)=0 \Rightarrow \log\frac{\det g'}{\det g} = f$$
$$\Rightarrow R(\omega_0(\varphi_0)) = \omega_0 \qquad \square$$

Prop 4.4.1 Openness of interval for Eqn (2)
Suppose Eqn (2) has a solution $\varphi_t \in \cancel{C^{k,\alpha}} \cap H^{k,\alpha}$ for $t=\tau$, $0\le\tau<1$.
Then φ_τ uniquely extends to $\{\varphi_t \mid t\in[0,1) \cap [\tau-\varepsilon,\tau+\varepsilon]\}$

Pf: $\Gamma(\varphi,t) := A(\varphi)+t\varphi+L(0,\varphi)-f$

$$D_\varphi\Gamma(\psi) = (\Delta_\varphi + t)\psi + \int \psi\,\omega_\varphi^n/V,$$ must show $D_\varphi\Gamma$ invertible at (φ_τ,τ)

Case 1: $\tau=0$: $D_\varphi\Gamma\big|_{(\varphi_0,0)}$ is $C^{k,\alpha}(X) \ni \psi \mapsto \Delta_\varphi\psi + \int_X \psi\,\omega_{\varphi_0}^n/V \in C^{k-2,\alpha}$

invertible by use of Green's fn ~~linear elliptic theory~~

Note: diff. b/w this and Siu's New (MA) eqn: $A(\phi) = -t\phi_t + F - \log\left(\frac{1}{V}\int e^{-t\phi_t+F}\omega_0^n\right)$

~~Siu ... this to~~ solutions of this are $\psi_t - \frac{1}{V}\int \psi_t \omega_0^n$ $\quad - \left(\frac{1}{V}\int e^F\omega_0^n\right)^{-1}\cdot\frac{1}{V}\int(-t\delta\phi\, e^{-t\phi_t+F}\omega_0^n)$

this has 0 avg so invertible at 0

$\|\nabla_\varphi \bar\partial f\|^2 =$

Case 2: $\tau \neq 0$

$$D_\varphi \Gamma|_{(\varphi_\tau,\tau)} = (\Delta_{\varphi_\tau} + \tau)\psi + \int \psi\, \omega_{\varphi_\tau}^n / V$$

$R(\varphi_\tau) - \tau\omega_0(\varphi_\tau)$ pos. def.

$$\cancel{g'^{i\bar j} R_{i\bar j}' - \tau g'^{i\bar j} g'_{i\bar j} = -\Delta_{\varphi_\tau}\left(\log \frac{\omega_0^n(\varphi_\tau)}{\omega_0^n}\right) - \tau n > 0}$$

$$\cancel{\Rightarrow (-\Delta_{\varphi_\tau})u > n\tau}$$

$$\Delta_\varphi f = -\tau f$$

$$\Rightarrow \Delta_\varphi \bar\partial f = -\tau\bar\partial f \quad \Rightarrow \quad \tau\|\bar\partial f\|^2 = \tau\int g^{i\bar j}\,\overline{\partial_{\bar i} f}\,\partial_{\bar j} f \det g =$$

$$= \int g^{i\bar j}\,\overline{\partial_{\bar i} f}\left(-\Delta_\varphi \partial_{\bar j} f\right)\det g$$

$$= (-\Delta_\varphi \bar\partial f, \bar\partial f)$$

$$= \|\bar\nabla_\varphi \bar\partial f\|^2 + (Ric_\varphi \bar\partial f, \bar\partial f)$$

BK Formula for $T^{*0,1}$

$(\bar\nabla\bar\partial f)_{\bar p} =$

$$\cancel{(-\Delta_\varphi\bar\partial f, \bar\partial f) = -\int g^{i\bar j}\,\overline{\partial_{\bar i} f}\left(g^{k\bar l}\partial_k\partial_{\bar l}\partial_{\bar j} f\right)\det g}$$

$$\cancel{= \int g^{i\bar j}\partial_k\overline{\partial_{\bar i} f}\, g^{k\bar l}\partial_{\bar l}\partial_{\bar j} f \det g + \int}$$

$$= -\int g^{i\bar j}\,\overline{\nabla_{\bar i} f}\, g^{k\bar l}\nabla_{\bar j}\nabla_k\nabla_{\bar l} f \det g + \int g^{i\bar j}\,\overline{\nabla_{\bar i} f}\, g^{k\bar l}[\nabla_k, \nabla_{\bar j}]$$

$Ric_\varphi > \tau \Rightarrow \bar\partial f = 0 \Rightarrow f \equiv \text{const} \Rightarrow \Delta_\varphi f = 0 \neq -\tau f \Rightarrow \tau$ smaller than lowest pos. eigenvalue of $-\Delta_\varphi$

$\Rightarrow D_\varphi\Gamma$ invertible

$$(-\Delta_\varphi \bar\partial f, \bar\partial f) = -\int g^{i\bar j}\,\overline{\partial_{\bar i} f}\left(g^{k\bar l}\partial_k\partial_{\bar l}\partial_{\bar j} f\right)\det g$$

$=$

FIRST INTRODUCE I & J

$$R = g_\varphi^{i\bar{j}} R_{i\bar{j}} = g_\varphi^{i\bar{j}}\left((1-t)(g_0)_{i\bar{j}} + t(g_\varphi)_{i\bar{j}}\right)$$
$$= g_\varphi^{i\bar{j}}\left((g_\varphi)_{i\bar{j}} - (1-t)\partial_i\partial_{\bar{j}}\varphi\right)$$

Eqn$^{(2)}$ $\log \dfrac{\det g_t^\#}{\det g_0} = -t\varphi_t + L(0,\varphi_t) + f$

Mabuchi functional $\mu(\omega^t) = M(\omega_0,\omega^t) := -\int_a^b \int_X \dot{\varphi}_t (R_\varphi - n)\, \omega_{\varphi_t}^n / V$

$$= +\int_a^b \int_X \dot{\varphi}_t (1-t)(\Delta_{\varphi_t}\varphi_t)\, \omega_{\varphi_t}^n / V$$

$\Rightarrow \dfrac{d}{dt}\mu(\omega^t) = \int_X (1-t)\dot{\varphi}_t (\Delta_{\varphi_t}\varphi_t)\, \omega_{\varphi_t}^n / V$

$= -(1-t)\dfrac{d}{dt}(I_t - J_t)$ $\longleftarrow$ formula in Adam's talk (or easily derived from) must introduce I & J

Diff. Eqn (2) wrt t : $\Delta_{\varphi_t}\dot{\varphi}_t + t\dot{\varphi}_t + \varphi_t + C_t = 0$

$\dfrac{d}{dt}(I_t - J_t) = -\int \varphi_t (\Delta_{\varphi_t}\dot{\varphi}_t)\, \omega_{\varphi_t}^n / V$ always

$= \int (\Delta_{\varphi_t}\dot{\varphi}_t + t\dot{\varphi}_t + C_t)(\Delta_{\varphi_t}\dot{\varphi}_t)\, \omega_{\varphi_t}^n / V$ here

$= \int (\Delta_{\varphi_t}\dot{\varphi}_t + t\dot{\varphi}_t)(\Delta_{\varphi_t}\dot{\varphi}_t)\, \omega_{\varphi_t}^n / V \geq 0$

since $t < \lambda_1(-\Delta_{\varphi_t})$

$\Rightarrow$ ~~Mabu~~ K-energy ~~decreasing~~ non-increasing in t along method of continuity for Eqn (2). $\square$

$$\int_0^1 (1-x)^{n-k-1} x^k \, dx = \begin{cases} \text{int by parts k times: } & \frac{1}{k+1}\binom{n}{k+1}^{-1} \\ \text{expand binomial: } & \sum_{j=0}^{n-k-1} (-1)^j \binom{n-k-1}{j} \frac{1}{j+k+1} \end{cases}$$

Def of I & J

- $I(\varphi', \varphi'') := \int (\varphi'' - \varphi')(\omega_0(\varphi')^n - \omega_0(\varphi'')^n)/V$

$\Rightarrow I(0, \varphi^*) =$ [illegible] in Siu

$\cdot\ I(\omega_\varphi, \omega_0)$

factor of V difference and arguments reversed, otherwise the same

- $J(\varphi', \varphi'') := -L(\varphi', \varphi'') + \int (\varphi'' - \varphi') \omega_0(\varphi')^n / V$

$$= -\int_a^b \left(\int \dot{\varphi}_t \, \omega_0(\varphi_t)^n / V \right) dt + \int (\varphi'' - \varphi') \omega_0(\varphi')^n / V$$

Siu: $J(\omega_\varphi, \omega_0) = J(\varphi, 0) = \int_0^1 \left(\int_X \varphi (\omega_0^n - \omega_{s\varphi}^n) \right) ds \quad = J(0, \varphi)$ in BM notation

<u>compare</u>: $I(0,\varphi)$ w/ $I(0,\varphi) - J(0,\varphi)$

$$V \cdot J = \int_0^1 \left(\int \varphi (\omega_0^n - \omega_{s\varphi}^n) \right) ds = -\int_0^1 \left(\sum_{j=0}^{n-1} \int \varphi \binom{n}{j+1} (si\partial\bar\partial\varphi)^{j+1} \omega_0^{n-j-1} \right) ds$$

$$= \int_0^1 \left(\sum_{j=0}^{n-1} \int_X (s\, i\partial\varphi \wedge \bar\partial\varphi) \binom{n}{j+1} (s\, i\partial\bar\partial\varphi)^j \omega_0^{n-j-1} \right) ds$$

$$= \sum_{j=0}^{n-1} \int (i\partial\varphi \wedge \bar\partial\varphi) \frac{1}{j+2} \binom{n}{j+1} \underbrace{(i\partial\bar\partial\varphi)^j}_{= (\omega_0(\varphi) - \omega_0)^j} \omega_0^{n-j-1}$$

$$= \sum_{j=0}^{n-1} \int (i\partial\varphi \wedge \bar\partial\varphi) \frac{1}{j+2} \binom{n}{j+1} \sum_{k=0}^{j} \binom{j}{k} (-1)^{j-k} \omega_0(\varphi)^k \omega_0^{j-k} \omega_0^{n-j-1}$$

$$\binom{n}{j-1} = \frac{n!}{(j-1)!(n-j+1)!}$$

$$= \sum_{k=0}^{n-1} \int (i\partial\varphi \wedge \bar\partial\varphi) \left(\sum_{j=k}^{n-1} (-1)^{j-k} \binom{j}{k} \binom{n}{j-1} \frac{1}{j+2} \right) \omega_0^{n-k-1} \omega_0(\varphi)^k$$

alg. manipulation of binomial

$$= \sum_{k=0}^{n-1} \int (i\partial\varphi \wedge \bar\partial\varphi) \left(\sum_{j=0}^{n-k-1} (-1)^j \binom{n-1}{k} \binom{n-k-1}{j} n \left(\frac{1}{j+k+1} - \frac{1}{j+k+2} \right) \right) \omega_0^{n-k-1} \omega_0(\varphi)^k$$

identity

$$= \sum_{k=0}^{n-1} \int (i\partial\varphi \wedge \bar\partial\varphi) \left(1 - \frac{k+1}{n+1} \right) \omega_0^{n-k-1} \omega_0(\varphi)^k$$

$$V \cdot I = \int \varphi (\omega_0^n - \omega_0(\varphi)^n) = \int \varphi (\overbrace{\omega_0 - \omega_0(\varphi)}^{-i\partial\bar\partial\varphi}) \sum \omega_0^{n-k-1} \omega_0(\varphi)^k$$

$$= \sum_{k=0}^{n-1} \int (i\partial\varphi \wedge \bar\partial\varphi) \omega_0^{n-k-1} \omega_0(\varphi)^k$$

$\Rightarrow \frac{1}{n+1} I \le J \le \left(1 - \frac{1}{m+1}\right) I \quad \Rightarrow \frac{1}{n+1} I \le I - J \le \frac{n}{n+1} I$

$I \ge 0$ since I is integral of sum of positive forms

Lower bound for Green's fn of Laplacian

(M, g) a cpt Riem. mfld of real dim n.
$\square$ the pos. Laplacian of (M, g) for functions
$\mathcal{H}_\nu$ the Hilbert space $W^{\nu,2} = L^2_\nu$

Skip

$$((1+\square)f, f)_0 \geq C\|f\|_1^2$$

Replace f by $(1+\square)^{-1}f$, get $C\|(1+\square)^{-1}f\|_1^2 \leq (f, (1+\square)^{-1}f)_0 \leq \|f\|_0 \|(1+\square)^{-1}f\|_0$
$\leq \|f\|_0 \|(1+\square)^{-1}f\|_1$

and $\|(1+\square)^{-1}f\|_1 \leq \frac{1}{\sqrt{C}}\|f\|_0$

$\Rightarrow (1+\square)^{-1} : L^2 \to L^2$ factors through $L^2_1 \hookrightarrow L^2_0$, therefore cpt

μ the eigenvalues of $(1+\square)^{-1}$

$\Rightarrow \lambda_i = \frac{1}{1+\mu_i}$, λ_i the eigenvalues of $\square$, $\{f_i\}$ an ON basis of L^2
$\square f_i = \lambda_i f_i$

Let $H(x,y,t) = \sum_{i=0}^{\infty} e^{-\lambda_i t} f_i(x) f_i(y)$, <u>heat kernel</u>

$$\begin{cases} \left(\frac{\partial}{\partial t} + \square\right) H(x,y,t) = 0 \\ \int_M H(x,y,0) f(y)\, dy = f(x) \end{cases},$$

$H(x,y,t)$ is everywhere positive since reproducing at 0, + ~~max~~ (min) principle arg.

Let $G(x,y) = \sum_{i=1}^{\infty} \frac{1}{\lambda_i} f_i(x) f_i(y)$, <u>Green's fn</u> $\left(\Rightarrow \int G(x,y)\,dy = 0\right.$ since f_i's $\perp f_0 = \text{const}\left.\right)$

$$f(x) = \frac{1}{\mathrm{Vol}\, M}\int_M f(y)\,dy + \int_M G(x,y)\, \square_y f(y)\, dy$$

$\Rightarrow G^+ = -(-G) \leq \frac{1}{}$

$-G(x,y,t) = \frac{1}{\mathrm{Vol}M} - H(x,y,t) < \frac{1}{\mathrm{Vol}M}$

Let $G(x,y,t) = \sum_{i=1}^{\infty} e^{-\lambda_i t} f_i(x) f_i(y) = H(x,y,t) - \frac{1}{\mathrm{Vol}\, M}$ $\quad G(x,y,t) > -\frac{1}{\mathrm{Vol}M}$

Then $G(x,y) = \int_{t=0}^{\infty} G(x,y,t)\,dt$ $\Rightarrow$ $\cancel{\int_M \int_0^\infty G(x,y,t)\,dt\,dV = 0 \stackrel{?}{=} \int_0^\infty \left(\int_M G(x,y,t)\,dV\right)dt}$
$\cancel{G(x,y,t) > -\frac{1}{\mathrm{Vol}M}}$

$\int G^+ - G^- = 0$, $-\int G^- > -1$

and since $H(x,y,t) > 0$, $\int_M |G(x,y,t)|\,dy = \int_M G^+ + G^- \leq 2$. ✓

Let $a_g := D_g^2 \cdot \operatorname{Inf}\{ Ric(V,V)/(n-1),\ \|V\|_g = 1\}$

Thm: Suppose $a_g \geq -\alpha^2$. Then $G(x,y) \geq -\gamma(n,\alpha)\frac{D_g^2}{V_g}$ (Cheng-Li)

$(\frac{\partial}{\partial t} + \square_x) G(x,y,t) = 0$

Note: (i) $\int_M |G(x,y,t)|\,dy \leq 2$

(ii) $\int_M G(x,y,t)\,dy = 0$

since $\int H(x,z,s)G(z,y,t)\,dz = G(x,y,t+s)$ & (ii)

$G(x,y,t+s) = \int_M G(x,z,s)G(z,y,t)\,dz$

$\therefore -\frac{\partial}{\partial t}G(x,x,t) = -\frac{\partial}{\partial t}\int_M G(x,z,\frac{t}{2})G(z,x,\frac{t}{2})\,dz$

pos. Laplacian

$= \int G(x,z,\frac{t}{2}) \cdot -\square_x G(x,z,\frac{t}{2})\,dz$

$= \int |d_z G(x,z,\frac{t}{2})|^2\,dz$

must use diff. arg for surfaces $(n=2)$

$L^{\frac{2n}{n-2}}$

$\geq C_S^2 \left(\int |G(x,z,\frac{t}{2})|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}$ (Sobolev inequality)

$C = K(n,\alpha)\frac{V_g^{1/n}}{D_g}$

~~$\int |G(x,z,\frac{t}{2})|^{\frac{2n}{n-2}} = \int |G(x,z,\frac{t}{2})| \cdot |G(x,z,\frac{t}{2})|^{\frac{n+2}{n-2}}$~~

$\geq$

Hölder

$\left(\int |G(x,z,\frac{t}{2})|^2\right)^r \leq \left(\int |G(x,z,\frac{t}{2})|^a |G|^b\right)^r$

$\leq \left(\int |G|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \left(\int |G|\right)^{\frac{4}{n}}$

$2 = a+b$

$ap = \frac{2n}{n-2}$ $\quad \frac{1}{p}+\frac{1}{q} = 1$ $\quad \frac{r}{p} = \frac{n-2}{n}$

$bq = 1$

$\frac{1}{p} + b = 1$

$a\frac{1}{1-b} = \frac{2n}{n-2}$ $\quad r = rb + \frac{n-2}{n}$

$a = \frac{2n}{n-2}(1-b)$ $\quad a - \frac{2n}{n-2}b = \frac{2n}{n-2}$ $\quad \Rightarrow \left(\frac{n-2}{n-2} - \frac{2n}{n-2}\right)b = 2 - \frac{2n}{n-2}$

$a+b = 2$ $\quad -\frac{(n+2)}{n-2}\,b = \frac{-4}{n-2}$

$b = \frac{4}{n+2}$

$\Rightarrow a = \frac{2n}{n+2} \Rightarrow p = \frac{n+2}{n-2},\ q = \frac{n+2}{4},\ r = \frac{n+2}{n}$ $\quad \frac{r}{q} = \frac{4}{n}$

$\geq 2^{-\frac{4}{n}} C_S^2 \left(\int |G|^2\right)^{\frac{n+2}{n}}$

$\left(\int G(x,z,\frac{t}{2})G(x,z,\frac{t}{2})\,dz\right)^{\frac{n+2}{n}}$

$= C^2\, G(x,x,t)^{\frac{n+2}{n}}$

~~and $\lim_{t\to\infty} G(x,x,t) \overset{?}{\geq} \ell$ for any finite ℓ~~

$\Rightarrow -G'(x,x,t)\,G(x,x,t)^{-\frac{(n+2)}{n}} \geq C^2$

$= \frac{d}{dt}\left(G(x,x,t)^{-\frac{2}{n}}\right)\cdot\frac{n}{2}$ $\quad \Rightarrow \frac{n}{2}G(x,x,t)^{-\frac{2}{n}} \geq C^2 t \Rightarrow G(x,x,t) \leq \left(\frac{2}{n}\right)^{-\frac{n}{2}} C^{-n} t^{-\frac{n}{2}}$

~~$|G(x,y,t)|^2 = \sum_{ij} e^{-(\lambda_i t + \lambda_j t)} f_i(x)f_i(y)f_j(x)f_j(y) \leq \sum_i e^{-\lambda_i \frac{t}{2}} f_i(x)f_i(x) \sum_j e^{-\lambda_j \frac{t}{2}}$~~

~~$\sum ab \leq \sqrt{\sum a^2}\sqrt{\sum b^2}$~~

$$e^{-\lambda_i \frac{t}{2}} f_i(x)\, e^{-\lambda_i \frac{t}{2}} f_i(y)$$

$$|G(x,y,t)| \le \Big|\sum_i e^{-\lambda_i t} f_i(x) f_i(y)\Big| \le \sqrt{\sum e^{-\lambda_i t} f_i(x) f_i(x)}\,\sqrt{\sum e^{-\lambda_i t} f_i(y) f_i(y)}$$

$$= \sqrt{G(x,x,t)}\,\sqrt{G(y,y,t)}$$

$$\le \left(\tfrac{2}{n}\right)^{-\frac{n}{2}} C^{-\frac{n}{2}} t^{-\frac{n}{2}}$$

$$G(x,y) = \int_0^\infty G(x,y,t)\,dt \ge \cancel{-\int_0^\infty \left(\tfrac{n}{2}\right)^{\frac{n}{2}} C^{-\frac{n}{2}} t^{-\frac{n}{2}}\,dt}$$

need bd for small t, but $G(x,y,t) \ge -\frac{1}{V_g}$

$$\ge -\int_0^\tau \frac{dt}{V_g} - \left(\tfrac{n}{2}\right)^{\frac{n}{2}} C^{-\frac{n}{2}} \int_\tau^\infty t^{-\frac{n}{2}}\,dt \qquad -\tfrac{2}{n-2}\, t^{\frac{2-n}{2}}$$

$$\ge -\frac{\tau}{V_g} - \left(\tfrac{n}{2}\right)^{\frac{n}{2}} C^{-\frac{n}{2}} \frac{2}{n-2} \cdot \tau^{-\frac{(n-2)}{2}}$$

Let $\tau = D_g^2 \Rightarrow G(x,y) \ge -\frac{D_g^2}{V_g} - \left(\tfrac{n}{2}\right)^{\frac{n}{2}} \frac{2}{n-2} K(n,\alpha)^{-n} \frac{D_g^n}{V_g} D_g^{-(n-2)}$

$$\ge -\gamma(n,\alpha)\frac{D_g^2}{V_g} \qquad \square$$

Remains to prove Sobolev inequality from Ric bd.

- relate Sobolev inequality to isoperimetric ineq.
- Ric bd $\Rightarrow$ bd on isoperimetric const

Gallot, Riem. Geom.

$a \geq -c$ $\quad a \leq c$

$b \geq -d$ $\quad b \geq -d$

ab $\quad \Rightarrow ab \geq$

Payoff: ~~Osc~~ Bd on Osc φ

$\omega_0(\varphi) = \omega_0 + i\partial\bar\partial\varphi \Rightarrow g'_{i\bar j} = g_{i\bar j} + \partial_i\partial_{\bar j}\varphi \Rightarrow -\Delta_0\varphi + n = Tr_{g_0} g' \geq 0 \Rightarrow -\Delta_0\varphi \leq n$

$-\Delta_\varphi \varphi + Tr_{g_\varphi} g_0 = n \Rightarrow -\Delta_\varphi\varphi \geq -2n$

~~$\varphi(x) = \frac{1}{V}\int \varphi\,\omega_0^n/n! + \int G_0(x,y)(-\Delta_0\varphi(y))\,\omega_0^n/n!$~~

~~$\geq \frac{1}{V}\int \varphi\,\omega_0^n/n! + \int(-\gamma(n,0)\frac{D_0^2}{V_0})($~~

~~and $\varphi(x) = \frac{1}{V}\int \varphi\,\omega_\varphi^n/n! + \int G_\varphi(x,y)(-\Delta_\varphi\varphi(x))\,\omega_\varphi^n/n!$~~

$$\varphi(x) = \frac{1}{V}\int \varphi\,\omega_0^n/n! + \int G_0(x,y)(-\Delta_0\varphi(y))\,\omega_0^n/n!$$
$$= \frac{1}{V}\int \varphi\,\omega_0^n/n! + \int (G_0(x,y)+K_0)(-\Delta_0\varphi(y))\,\omega_0^n/n!\,, \quad K_0 \text{ const st } G_0(x,y)+K_0 \geq 0$$

$$\leq \frac{1}{V}\int \varphi\,\omega_0^n/n! + \int (G_0(x,y)+K_0)(+2n)\,\omega_0^n/n!$$
$$= \frac{1}{V}\int \varphi\,\omega_0^n/n! + 2n(K_0)V$$

and also

$$\varphi(x) = \frac{1}{V}\int \varphi\,\omega_\varphi^n/n! + \int (G_\varphi(x,y)+K_\varphi)(-\Delta_\varphi\varphi(y))\,\omega_\varphi^n/n!$$

$$\geq \frac{1}{V}\int \varphi\,\omega_\varphi^n/n! + \int (G_\varphi+K_\varphi)2n\,\omega_\varphi^n/n!$$

$$\leq \frac{1}{V}\int \varphi\,\omega_\varphi^n/n! + 2nK_\varphi V$$

$$\Rightarrow \operatorname{Osc}\varphi_t \leq \frac{1}{V}\int \varphi(\omega_0^n/n! - \omega_\varphi^n/n!) + 2nV(K_\varphi+K_0)$$

$$\leq I(0,\varphi_t) + 2nV(K_{\varphi_t}+K_0) \leq \boxed{I(0,\varphi_t) + C(n)(K_0 + \tfrac{1}{t})}$$

$Ric_{\varphi_t} \geq t\omega_\varphi$ along Method of Cont. $\Rightarrow$ by Meyer's thm, $D_{g_t} \leq \pi\left(\frac{n-1}{t}\right)^{1/2}$

$$VK_\varphi \leq +\gamma(2n,0)\frac{D_g^2}{V} \geq +\beta(n)\pi^2\left(\frac{n-1}{t}\right) \geq +\frac{C(n)}{t}$$

(Closedness ~~#~~ back to 0)

Thm: Let $0<\tau<1$, φ_τ any solution of Eqn (2) $(A(\varphi_t) = -t\varphi_t - L(0,\varphi_t) + f)$

Then φ_τ extends uniquely to a smth family $\{\varphi_t \mid 0 \le t \le \tau\}$ of solutions.

In particular (since φ_0 unique), Eqn (2) has unique solution at τ.

Pf:

Step 1: ~~We have~~ (Assume such an extension is not possible beyond $\{\varphi_t \mid \sigma < t \le \tau, \sigma > 0\}$)

We have $0 \le I_t \le (n+1)(I_t - J_t) \le (n+1)(I_\tau - J_\tau)$

Let $F_t := -t\varphi_t - L(0,\varphi_t) + f \in C^\infty(X,\mathbb{R})$

$\int \omega_0^n = \int \exp(A(\varphi_t))\omega_0^n = \int_X \exp(F_t)\,\omega_0^n \Rightarrow \exists\, x_t \in X$ st $F_t(x_t) = 0$

$\therefore \forall x \in X,\ |F_t(x)| = |F_t(x) - F_t(x_t)| = |-t(\varphi_t(x) - \varphi_t(x_t)) + f(x) - f(x_t)|$

$\le t \operatorname{Osc}\varphi_t + 2\|f\|_{C^0}$

$\le tI_t + 2n(tK_0V_0 + (n-1)\beta(n)\pi^2) + 2\|f\|_{C^0}$

$\Rightarrow \|F_t\|_{C^0} \le K_1$

Then the M-A eqn $A(\varphi_t) = F_t \Rightarrow \operatorname{Osc}\varphi_t \le K_2$ (Do we have C^0 bd at this pt?)

From constraint

$\int e^{-t\varphi_t - L(0,\varphi_t)}$

Once we have C^0 bd, C^2 & $C^{2,\alpha}$ bds as before. □

$H^0(T^{*(0,1)}) = \overline{H^0(T^{*(1,0)})} = \overline{H^0(T^{1,0})}$

$H^0(K$

$$\boxed{\text{Remains only to prove openness at } t=1, \text{ which is where hol. vf's and orbits come in.}}$$

Eqn(1): $A(\varphi_t) = -t\varphi_t + f$, $\quad \Gamma(\varphi_t) = A(\varphi_t) + t\varphi_t - f$

Eqn(2): $A(\varphi) = -t\varphi_t - L(0,\varphi_t) + f$

$$D_{\varphi_t}\Gamma(t,\varphi_t)\Big|_{t=1} \cdot \dot\varphi = (\Delta_{\varphi_t} + 1)\dot\varphi_t$$

$(\nabla_{\bar j}\bar\partial f)_{\bar l} = \partial_{\bar j}\partial_{\bar l} f - \Gamma^{\bar m}_{\bar j \bar l}\partial_{\bar m} f$

$= \partial_{\bar j}\partial_{\bar l} f - g^{\bar m p}\partial_{\bar j} g_{p\bar l}\partial_{\bar m} f$

$= 0$

$\partial_{\bar l}(g^{i\bar j}\partial_{\bar j} f) = -g^{i\bar q} g^{p\bar j}\partial_{\bar l} g_{p\bar q}\partial_{\bar j} f + g^{i\bar j}\partial_{\bar l}\partial_{\bar j} f$

$g_{k\bar i}$

$= 0$

$\Rightarrow \partial_{\bar l}\partial_{\bar k} f - g^{p\bar j}\partial_{\bar l} g_{p\bar k}\partial_{\bar j} f = 0$

$(\Delta_{\varphi_t} + 1)f = 0 \iff \overline{\nabla}_\varphi \bar\partial f = 0$ by BK formula $\quad t\|\bar\partial f\|^2 = (-\Delta_\varphi \bar\partial f, \bar\partial f) = \|\overline{\nabla}_\varphi \bar\partial f\|^2 + (Ric_\varphi \bar\partial f, \bar\partial f)$

$\Rightarrow g'^{i\bar j}\partial_{\bar j} f$ is holomorphic

Conversely, if $X = X^i \frac{\partial}{\partial z^i}$ any hol. $(1,0)$-vf, then $g_{\bar j i} X^i d\bar z^j$ is $\bar\partial$-closed

Hence $\bar\partial$-exact since anticanonical bundle pos

$\Rightarrow X = \uparrow \bar\partial f$

$H^0(T^{*0,1}) \cong H^{0,1} \cong H^1(\Theta) \cong \cancel{H^1(\Theta(K_M^{-1} \otimes K_M)) \cong H^{n-1}(K_M)}$

know $H^q(K_M^{-1}) = 0$ for $q \geq 1$, $\quad H^q(K_M^{-p}) \cong H^{n-q}(K_M^{p+1})$

$H^1(K_M^{*}) = H^{n-1}($

$H^1(\Theta) = H^{n-1}(K_M) \cong$

$\therefore (\Delta_{\varphi_1} + 1)f = 0$ is invertible iff X has no hol. vf's.

- <u>Suppose X has hol. vfs</u>, let $\theta \in \mathcal{E}$, ie θ KE metric.

Let $\mathfrak{g} = H^0(X, \mathcal{O}(TX))$. $Y \in \mathfrak{g}$.

Let $Y_{\mathbb{R}} = Y + \overline{Y}$, $\quad \mathfrak{g}_{real} = \{Y_{\mathbb{R}} \mid Y \in \mathfrak{g}\}$ $\quad Y \to Y_{\mathbb{R}}$ is an iso $(\mathfrak{g}, \sqrt{-1}) \cong (\mathfrak{g}_{real}, \underset{\text{cx structure}}{J})$ of cx Lie algebras

G-orbit $\mathcal{O}$ through θ: $\mathcal{O} \cong G/K_\theta$, K_θ isotropy subgp

Let $\mathfrak{k}_\theta$ be the set of all Killing vfs on X wrt θ, regarded as element of

$\mathfrak{k}_\theta$ is the Lie subalg of $\mathfrak{g}$ corr. to K_θ in G

$$\tilde{Y} = \frac{1}{2}\theta^{\bar\beta\alpha}\partial_{\bar\beta}\varphi\frac{\partial}{\partial z^\alpha} + \overline{\frac{1}{2}\theta^{\bar\beta\alpha}\partial_{\bar\beta}\varphi\frac{\partial}{\partial z^\alpha}} = \frac{1}{2}\theta^{\bar\beta\alpha}\partial_{\bar\beta}\varphi\frac{\partial}{\partial z^\alpha} + \frac{1}{2}\theta^{\bar\alpha\beta}\partial_\beta\varphi\frac{\partial}{\partial \bar z^\alpha}$$

$$\boxed{\frac{\partial}{\partial t}\left((y^\varphi_{\theta t})^*\theta\right)\Big|_{t=0}} = L_{\tilde Y}\theta = (d\circ\iota_{\tilde Y} + \iota_{\tilde Y}\circ d)\theta = d\circ\iota_{\tilde Y}\theta = \frac{i}{2} d\left(\frac{1}{2}\theta^{\bar\beta\alpha}\partial_{\bar\beta}\varphi\,\theta_{\alpha\bar\gamma}\,d\bar z^\gamma + \frac{1}{2}\theta^{\bar\alpha\beta}\partial_\beta\varphi\,\theta_{\gamma\bar\alpha}\,dz^\gamma\right)$$

$$= \frac{i}{2} d\left(\partial_{\bar\gamma}\varphi + -\partial_\gamma\varphi\right)$$

$$\boxed{= i\partial\bar\partial\varphi} \qquad \boxed{\text{Just state}}$$

$f(z_1,\dots,z_n) = (w_1,\dots,w_n)$

$f^*\omega = \cancel{g_{i\bar j}(f(z))\frac{\partial f}{\partial z^i}dz^i\wedge\frac{\partial \bar f}{\partial \bar z^j}d\bar z^j}$

$\frac{i}{2}\sum g_{i\bar j}(f(z))\, d(f(z))^i\wedge d\overline{(f(z))^j} = \sum g_{i\bar j}(\partial$

For each $\varphi \in C^\infty(X,\mathbb{C})$, define the vector field

$$Y^\varphi_\theta := \frac{1}{2}\theta^{\bar\beta\alpha}\partial_\beta\varphi\frac{\partial}{\partial z^\alpha}, \qquad \& \text{ 1-ps } y^\varphi_{\theta t} := \exp(tY^\varphi_{\theta\,\mathbb{R}}),\ t\in\mathbb{R}$$

<u>Thm (2.2)</u> Let $H_\theta := \{\varphi\in C^\infty(X,\mathbb{R}) \mid (\Delta_\theta + 1)\varphi = 0\}$
set $\mathfrak{p}_\theta := \sqrt{-1}\,\mathfrak{k}_\theta$ and $H^{\mathbb{C}}_\theta := H_\theta\otimes_{\mathbb{R}}\mathbb{C}\subset C^\infty(X,\mathbb{C})$

Then
- $\mathfrak{k}_\theta = \{Y^\varphi_\theta \mid \varphi\in\sqrt{-1}H_\theta\}$ and $\mathfrak{p}_\theta = \{Y^\varphi_\theta \mid \varphi\in H_\theta\}$
- $\varphi\in H^{\mathbb{C}}_\theta \mapsto Y^\varphi_\theta \in \mathfrak{g}$ defines an isomorphism $H^{\mathbb{C}}_\theta \cong \mathfrak{g}$ $\Rightarrow \mathfrak{g} = \mathfrak{k}_\theta + \mathfrak{p}_\theta$, $\mathfrak{k}_\theta\cap\mathfrak{p}_\theta = \{0\}$

$\therefore\ T_\theta(\mathcal{O}) \cong T_e(G/K_\theta) = \mathfrak{p}_\theta \cong H_\theta$

$$\frac{\partial}{\partial t}\left((y^\varphi_{\theta t})^*\theta\right)\Big|_{t=0} \Leftrightarrow \quad Y^\varphi_\theta \longleftrightarrow \varphi$$

<u>Idea of pf of Main Thm:</u> If have 2 ~~orbits~~ (only one orbit) $\mathcal{O}_\nu$, $\nu=1,2$,
find elements $\theta_\nu = \omega + i\partial\bar\partial\lambda_{\theta_\nu}$ of $\mathcal{O}_\nu$ st (MA) $\frac{(\omega+i\partial\bar\partial\varphi)^n}{\omega^n} = \exp(-t\varphi + F_\omega)$ for some ω
has openness at $t=1$,
and $\varphi = \lambda_{\theta_\nu}$

Then $\theta_1 = \theta_2$, by uniqueness along method of continuity
θ_ν will be where $\inf I(\omega,\theta) - J(\omega,\theta)$ achieved

Let $\theta\in\mathcal{E}$, a KE metric
$\theta = \omega_0(\lambda_\theta)$, $\quad A(\lambda_\theta) = -\lambda_\theta + f \qquad \mathcal{O}\ni\theta \longleftrightarrow \lambda_\theta\in\bar{\mathcal{O}}$

For each $\varphi\in H_\theta$, have 1ps $y^\varphi_{\theta t}$, put $\theta(t) = (y^\varphi_{\theta t})^*\theta$, $\lambda(t) := \lambda_{\theta(t)}$
Then $\omega_0(\lambda(t)) = \theta(t) \Rightarrow \dot\lambda(0) = \varphi + C$ for some $C\in\mathbb{R}$
but also
$\Delta_\theta\dot\lambda(0) = -\dot\lambda(0)$ from MA eqn $\Rightarrow \int_X \dot\lambda(0)\theta^n = 0 = \int\varphi\theta^n \Rightarrow \dot\lambda(0) = \varphi$

$$\Rightarrow \begin{array}{ccccc} T_{\lambda_\theta}(\bar{\mathcal{O}}) & \longleftrightarrow & T_\theta(\mathcal{O}) & \longleftrightarrow & H_\theta \\ \dot\lambda(0) = \varphi & \longleftrightarrow & \dot\theta(0) = i\partial\bar\partial\varphi & \longleftrightarrow & \varphi. \end{array}$$

Lemma 6.2: Let $\iota : \mathcal{O} \to \mathbb{R}$ be the functional $\iota(\theta) = \iota(\omega_0, \theta) = I(\omega_0, \theta) - J(\omega_0, \theta)$
ι is proper, and so its min. is attained at some pt of the orbit $\mathcal{O}$.

Pf. ~~Must show~~ We know $I - J \geq 0$
Must show convergence of seq. of $\theta_i \in \mathcal{O}$ approaching $\inf \iota(\theta)$
Use regularity of solutions of $A(\psi_i) = -\psi_i + f$.

We will have unif $C^{2,\alpha}$ estimates on ψ_i, hence unif bds on $\theta_i = \omega(\psi_i)$, hence cvg of subseq.
if can get C^0 bds
If $|\iota(\omega_0(\psi))| \leq r$, then $I(0,\psi) \leq (n+1) r$
$\Rightarrow$ as before, $\operatorname{Osc}\psi \leq K = (n+1) r + 2n\left(K_0 V_0 + (n-1)\underset{\gamma(2n,0)}{\beta(n)}\pi^2\right)$

and $\int \omega_0^n = \int \exp(A(\psi))\, \omega_0^n = \int \exp(-\psi + f)\, \omega_0^n$
$\Rightarrow \exists$ pt where $\psi(x) = f(x)$
$\Rightarrow \|\psi_\theta\|_{C^0} \leq K + \|f\|_{C^0}$ □

Lemma 6.3: Let $\theta \in \mathcal{O}$. ~~TFAE~~
~~(i)~~ If θ is a critical pt for ι
(ii) then $\int_X \lambda_\theta \varphi\, \theta^n = 0 \quad \forall \varphi \in H_\theta$
~~(iii)~~

Pf. Fix any $\varphi \in H_\theta$. w/ corr. 1-param families $\theta(t)$, $\lambda(t)$

$$\frac{d}{dt}\iota(\theta(t))\Big|_{t=0} = \frac{d}{dt}\left(I - J(0, \lambda(t))\right)\Big|_{t=0} = -\int \lambda_\theta\, \Delta_\theta(\dot\lambda(0))\, \theta^n / V = -\int \lambda_\theta\, \Delta_\theta \varphi\, \theta^n / V$$

$$= \int \lambda_\theta \varphi\, \theta^n / V.$$

Lemma 6.4: $\theta \in \mathcal{O}$ a critical pt for ι. Then $(\mathrm{Hess}\,\iota)_\theta$ is given by

$$(\mathrm{Hess}\,\iota)_\theta(\varphi',\varphi'') = \int_X \left(1 + \tfrac{1}{2}\Delta_\theta \lambda_\theta\right)\varphi'\varphi''\,\theta^n/V, \qquad \varphi',\varphi'' \in H_\theta$$

Pf: Let $\lambda_{s,t}$ be a 2 param. smth family of fns in $\overline{\mathcal{O}}$

st (i) $\lambda_{0,0} = \lambda_\theta$

(ii) $\frac{\partial}{\partial s}\lambda_{s,t}\big|_{0,0} = \varphi'$

(iii) $\frac{\partial}{\partial t}\lambda_{s,t}\big|_{0,0} = \varphi''$

$$A(\lambda_{s,t}) + \lambda_{s,t} - f = 0 \quad \Rightarrow \quad \underbrace{\Delta_{s,t}\left(\tfrac{\partial}{\partial t}\lambda_{s,t}\right)}_{=\,\theta_{s,t}^{i\bar{j}}\partial_i\partial_{\bar{j}}\left(\frac{\partial}{\partial t}\lambda_{s,t}\right)} + \frac{\partial}{\partial t}\lambda_{s,t} = 0 \quad \text{as before}$$

diff wrt s:

$$(\Delta_{s,t}+1)\left(\frac{\partial^2}{\partial s\partial t}\lambda_{s,t}\right) + -\theta_{s,t}^{i\bar{q}}\theta_{s,t}^{p\bar{j}}\,\partial_s\overbrace{(\theta_{s,t})_{p\bar{q}}}^{=\,(g_0)_{p\bar{q}} + \partial_p\partial_{\bar{q}}\lambda_{s,t}}\,\partial_i\partial_{\bar{j}}\left(\frac{\partial}{\partial t}\lambda_{s,t}\right)$$

$$= \quad '' \quad -\theta^{i\bar{q}}\theta^{p\bar{j}}\,\partial_p\partial_{\bar{q}}\left(\tfrac{\partial}{\partial s}\lambda\right)\partial_i\partial_{\bar{j}}\left(\tfrac{\partial}{\partial t}\lambda_{s,t}\right)$$

$$= \quad '' \quad -\left\langle \partial\bar{\partial}\left(\tfrac{\partial}{\partial s}\lambda\right), \partial\bar{\partial}\left(\tfrac{\partial}{\partial t}\lambda\right)\right\rangle_{s,t}$$

$$\Rightarrow (\Delta_\theta + 1)\left(\frac{\partial^2}{\partial s\partial t}\lambda_{s,t}\right)\Big|_{0,0} = \langle \partial\bar{\partial}\varphi', \partial\bar{\partial}\varphi''\rangle_\theta$$

Lemma 2.3: Let $\langle\ ,\ \rangle_\theta$ be the pairing of p-forms, θ a Kähler metric. Then $\forall\, \varphi, \psi \in H_\theta$ and all $\xi \in C^\infty(X,\mathbb{R})$

we have $$\Delta_\theta\langle\partial\xi,\partial\psi\rangle = \langle\partial\bar{\partial}\xi,\partial\bar{\partial}\psi\rangle + \langle\partial(\Delta_\theta\xi),\partial\psi\rangle$$

~~Thm part~~

Pf: Choose coords centered at x where $\theta_{\alpha\bar{\beta}} = \delta_{\alpha\beta}$, $d\theta_{\bar{\alpha}\beta}(x) = 0 \ \forall\, \alpha,\beta$

Note $\psi^\alpha_{\ \bar{\beta}} = 0$ since $\uparrow\bar{\partial}\psi$ is hol.

$$\Delta_\theta\langle\partial\xi,\partial\psi\rangle = \Delta_\theta(\xi_\alpha\theta^{\alpha\bar{\beta}}\psi_{\bar{\beta}}) = \Delta_\theta(\xi_\alpha\psi^\alpha) = \xi_{\alpha\bar{\beta}}\psi^\alpha_{\ \beta} + \xi_{\alpha\beta\bar{\beta}}\psi^\alpha$$
$$= \langle\partial\bar{\partial}\xi,\partial\bar{\partial}\psi\rangle_\theta + \langle\partial(\Delta_\theta\xi),\partial\psi\rangle$$

In particular ~~Also~~ $$(\Delta_\theta+1)\frac{\partial^2}{\partial s\partial t}\lambda_{s,t}\Big|_{0,0} = \langle\partial\bar{\partial}\psi',\partial\bar{\partial}\psi''\rangle \Rightarrow \frac{\partial^2}{\partial s\partial t}\lambda_{s,t}\Big|_{0,0} = \langle\partial\psi',\partial\psi''\rangle\ (\mathrm{mod}\ H_\theta)$$

$$\partial_\alpha \partial_{\bar\beta} \psi = \ldots \theta^{\alpha\bar\beta}\partial_{\bar\beta}\psi \quad \theta^{\alpha\bar\beta}\partial_\beta\overline{\psi} = \overline{\theta^{\bar\alpha\beta}\partial_{\bar\beta}\psi}$$

$$\ldots \theta^{\alpha\bar\beta}\partial_\alpha\psi\,\partial_{\bar\beta}\psi \qquad \psi$$

$\varphi - \Sigma\,\partial_\alpha\partial_{\bar\beta}\psi$

$$\int \left(\varphi(-\Delta_\theta\psi) - \theta^{\alpha\bar\beta}\partial_\alpha\varphi\,\overline{\partial_\beta\psi}\right)\xi\,\theta^n = \int \varphi\,\theta^{\alpha\bar\beta}\partial_\alpha\xi\,\partial_{\bar\beta}\psi + \theta^{\alpha\bar\beta}\partial_\alpha\varphi\,\partial_{\bar\beta}\psi\,\xi - \theta^{\alpha\bar\beta}\partial_\alpha\varphi\,\overline{\partial_\beta\psi}\,\xi$$

since ψ real

Let $\xi = (\Delta_\theta + 1)\zeta$

Also $\int(\varphi\psi - \langle\partial\varphi,\partial\psi\rangle)\xi\,\theta^n = -i\int(\varphi\,\partial\bar\partial\psi + \partial\varphi\wedge\bar\partial\psi)\,\xi\wedge n\theta^{n-1}$

$$= -i\int \xi\,\partial(\varphi\bar\partial\psi)\wedge n\theta^{n-1} \quad \longleftarrow \quad ?$$

$$= -i\int \varphi\,\partial\xi\wedge\bar\partial\psi\wedge n\theta^{n-1}$$

$$= \int\varphi\langle\partial\xi,\partial\psi\rangle\theta^n$$

$$= \int\varphi\left(\Delta_\theta\langle\partial\zeta,\partial\psi\rangle - \langle\partial\bar\partial\zeta,\partial\bar\partial\psi\rangle\right)\theta^n \quad \text{by Lemma}$$

$$+ \int\varphi\langle\partial\zeta,\partial\psi\rangle\theta^n$$

$$= -\int\varphi\langle\partial\bar\partial\zeta,\partial\bar\partial\psi\rangle\theta^n.$$

$$I - J(\omega_0, \theta_{s,t})$$

$$\therefore (\mathrm{Hess}\,\iota)(\varphi',\varphi'') = \frac{\partial^2}{\partial s\,\partial t}\left(I(0,\lambda_{s,t}) - J(0,\lambda_{s,t})\right)\Big|_{0,0}$$

$$= -\frac{\partial}{\partial s}\left[\int \lambda_{s,t}\Delta_{s,t}\left(\frac{\partial}{\partial t}\lambda_{s,t}\right)\theta^n_{s,t}/V\right]\Big|_{0,0}$$

$$= \frac{\partial}{\partial s}\int\lambda_{s,t}\left(\frac{\partial}{\partial t}\lambda_{s,t}\right)\theta^n_{s,t}/V\Big|_{0,0}$$

$$= \int\left(\varphi'\varphi'' + \lambda_\theta\left(\frac{\partial^2}{\partial s\,\partial t}\lambda_{s,t}\right)\Big|_{0,0} + \lambda_\theta\varphi''\Delta_\theta\varphi'\right)\theta^n/V$$

$$= \int \varphi'\varphi'' + \tfrac{1}{2}\left(\langle\partial\varphi',\partial\varphi''\rangle + \langle\partial\varphi'',\partial\varphi'\rangle\right)\lambda_\theta - \lambda_\theta\varphi'\varphi''\Big)\theta^n/V \quad \text{by L.2.3}$$

$$= \int\Big\{\varphi'\varphi'' + \tfrac{1}{2}\left((\Delta_\theta\varphi')\varphi'' + (\Delta_\theta\varphi'')\varphi' + \langle\partial\varphi',\partial\varphi''\rangle + \langle\partial\varphi'',\partial\varphi'\rangle\right)\lambda_\theta\Big\}\theta^n/V$$

$$= \int\left(\varphi'\varphi'' + \tfrac{1}{2}\lambda_\theta\Delta_\theta(\varphi'\varphi'')\right)\theta^n/V = \int\left(1 + \tfrac{1}{2}\Delta_\theta\lambda_\theta\right)\varphi'\varphi''\,\theta^n/V. \quad \square$$

Deal w/ $t=1$ by splitting up the space of fns

Fix $0<\alpha<1$,

let $H^{\perp}_{\theta,k} := \{\psi \in C^{k,\alpha}(X,\mathbb{R}) \mid \int \varphi\psi\theta^n = 0 \quad \forall \varphi \in H_\theta\}$

θ KE $\Rightarrow$ we have (i) $\theta = \omega_0(\lambda_\theta)$

(ii) $A(\lambda_\theta) = -\lambda_\theta + f$

(iii) $\lambda_\theta \in H^{\perp}_{\theta,k}$

Let $k \geq 2$

$\Phi : \mathbb{R}\times C^{k,\alpha} \to C^{k-2,\alpha}$, $(t,u) \mapsto \Phi(t,u) = A(u) + tu - f$

Note that any $v \in \mathcal{H}^{k,\alpha}$ satisfying $\Phi(t,v)=0$ is automatically in $\mathcal{H}$ ($= C^\infty$ ω_0-psh)

Let $P : C^{0,\alpha} (\cong H_\theta \oplus H^{\perp}_{\theta,0}) \to H_\theta$ be the proj.

For each $u \in C^{k,\alpha}$, $u = \lambda_\theta + \varphi + \psi$, $\varphi := P(u-\lambda_\theta) \Rightarrow \psi = (1-P)(u-\lambda_\theta)$

$$\Phi(t,u)=0 \iff \begin{cases} P\,\Phi(t,\lambda_\theta+\varphi+\psi) = 0 \\ \Psi(t,\varphi,\psi) = 0 \end{cases}$$

where

$\Psi : \mathbb{R}\times H_\theta \times H^{\perp}_{\theta,k} \to H^{\perp}_{\theta,k-2}$ def. by $\Psi(t,\varphi,\psi) := (1-P)\,\Phi(t,\lambda_\theta+\varphi+\psi)$

then $\Psi(t,0,0)=0$, $(D_\psi\Psi|_{(1,0,0)})\psi' = (\Delta_\theta+1)\psi' \in H^{\perp}_{\theta,k-2}$

which is invertible

$\Rightarrow$ for each fixed φ, can find unique solutions $\psi_{t,\varphi}$ to $\Psi(t,\varphi,\psi)$ in nbhd of 1

- $\psi_{1,0}=0$
- $\|\psi_{t,\varphi}\|_{C^{k,\alpha}} \leq \delta$ on U a nbhd of $(1,0)\in\mathbb{R}\times H_\theta$
- $\Psi(t,\varphi,\psi_{t,\varphi})=0$

Now diff. in t: $(\Delta_\theta+1)\left(\frac{\partial}{\partial t}\psi_{t,\varphi}|_{(1,0)}\right) = -\lambda_\theta$

and φ: $(D_\varphi\psi_{t,\varphi})|_{(1,0)}\varphi' = 0$ $\quad (D_\varphi\psi_{t,\varphi})|_{(1,0)} : H_\theta \to H^{\perp}_{\theta,k}$

$\Rightarrow$ on a small nbhd of λ_θ, eqn $\Phi(t,u) = A(u)+tu-f=0$, $u=\lambda_\theta+\varphi+\psi_{t,\varphi}$

reduces to $\Phi_0(t,\varphi)=0$, $\Phi_0(t,\varphi) := P\,\Phi(t,\lambda_\theta+\varphi+\psi_{t,\varphi})$

$\varphi, \psi \in H_\theta$, $\xi \in C^\infty$

$$\Delta_\theta \langle \partial \xi, \partial \psi \rangle = \langle \partial\bar\partial \xi, \partial\bar\partial\psi \rangle + \langle \partial(\Delta_\theta \xi), \partial\psi \rangle$$

$$-\int \varphi \langle \partial\bar\partial\xi, \partial\bar\partial\psi \rangle \theta^n = \int (\varphi\psi - \langle \partial\varphi, \partial\psi \rangle)(\Delta_\theta + 1)(\xi) \cdot \theta^n.$$

Now $\Phi(1, u) = 0$ for all $u \in \bar{O}$

Hence $\Phi_0 = 0$ on $\{t = 1\}$, so def $\Phi_1(t, \varphi) := \frac{1}{t-1}\Phi_0(t,\varphi)$

$$\Phi_1(1,0) = \frac{\partial \Phi_0}{\partial t}(1,0) \quad \text{by def.}$$

$$= \frac{\partial}{\partial t} P\left(A(\lambda_\theta + \varphi + \psi_{t,\varphi}) + t(\lambda_\theta + \varphi + \psi_{t,\varphi}) - f \right)\Big|_{(t,\varphi) = (1,0)}$$

$$= P\Big(\Delta_\theta\big(\tfrac{\partial}{\partial t}\psi_{t,\varphi}|_{(1,0)}\big) + \underbrace{\lambda_\theta + (\psi_{t,\varphi} + \varphi)\big|_{(1,0)}}_{\omega_0(\lambda_0 + \varphi + \psi_{t,\varphi})|_{(1,0)}} + 1 \cdot \overbrace{\big(\tfrac{\partial}{\partial t}\psi_{t,\varphi}\big)}^{\frac{\partial}{\partial t}\log\det \;\; g^{--}\partial_t g_{--}} \Big)$$

$$= 0$$

<u>Lemma 7.2</u>: $\forall\, \varphi', \varphi'' \in H_\theta$ where θ is a critical pt of ι,

$$\left(D_\varphi \Phi_1|_{(1,0)}(\varphi'), \varphi''\right)_{L^2(X,\theta)} = \int (1 + \tfrac{1}{2}\Delta_\theta\lambda_\theta)\varphi'\varphi''\, \theta^n/n!$$

$$= V_0\, (\mathrm{Hess}\, \iota)_\theta(\varphi', \varphi'')$$

(nice coincidence; geometric explanation?)

<u>Pf</u>:

$$D_\varphi \Phi_1|_{(1,0)}(\varphi') = \left(D_\varphi \tfrac{\partial}{\partial t}\Phi_0\right)\Big|_{(1,0)}(\varphi')$$

$$= \varphi' - P\underbrace{\langle \partial\bar\partial\big(\tfrac{\partial}{\partial t}\psi_{t,\varphi}|_{(1,0)}\big), \partial\bar\partial\varphi' \rangle}$$

$(\Phi_0 = (1-P)\Psi(t, \varphi, \psi_{t,\varphi}))$

$$D_\varphi\ \theta^{\alpha\bar\beta}\partial_\alpha\partial_{\bar\beta}(\dot\psi_{(1,0)}) = -\theta^{\alpha\bar\nu}\theta^{\mu\bar\beta} D_\varphi\theta_{\mu\bar\nu}\,\partial_\alpha\partial_{\bar\beta}\dot\psi_{(1,0)}$$

$$= -\theta^{\alpha\bar\nu}\theta^{\mu\bar\beta}\partial_\mu\partial_{\bar\nu}\varphi'\,\partial_\alpha\partial_{\bar\beta}\dot\psi$$

$$\Rightarrow \left(D_\varphi\Phi_1|_{(1,0)}\varphi', \varphi''\right)_{L^2} = \int \left[\varphi'\varphi'' - \varphi''\langle \partial\bar\partial(\dot\psi_{t,\varphi}|_{(1,0)}), \partial\bar\partial\varphi' \rangle\right]\theta^n/n!$$

$$= \int \left[\varphi'\varphi'' + (\varphi''\varphi' - \langle \partial\varphi'', \partial\varphi' \rangle)(-\lambda_\theta)\right]\theta^n/n!$$

$$= \int \left[\varphi'\varphi''(1 - \lambda_\theta) + \tfrac{1}{2}\lambda_\theta\left(\langle \partial\varphi', \partial\varphi'' \rangle + \langle \partial\varphi'', \partial\varphi' \rangle\right)\right]\theta^n/n!$$

$$= \int (1 + \tfrac{1}{2}\Delta_\theta\lambda_\theta)\varphi'\varphi''\,\theta^n/n! \quad \text{as before.} \quad (V = n!\,V_0)$$

$\square$

$\therefore$ If at θ, $(\mathrm{Hess}\,\iota)_\theta : H_\theta \times H_\theta \to \mathbb{R}$ is a non-degen bilinear form,

then $D_\varphi \Phi_1|_{(1,0)}$ is invertible

$\Rightarrow$ implicit fn thm ~~shows~~ gives $\Phi_1(t,\varphi)=0$ in φ is uniquely solvable in a nbhd of $(1,0)$

$\varphi(1)=0, \quad \Phi_1(t,\varphi(t))=0 \quad (1-\varepsilon \le t \le 1)$

$\Rightarrow$ $\Psi_t := \lambda_\theta + \varphi(t) + \Psi_{t,\varphi(t)}$ a smth 1-param family of solutions

From method of continuity, it will then follow that can extend all the way back to initial unique Ψ_0

<u>Pf of Main Thm:</u>

Fix $\tilde{\omega} \in \mathcal{K}^+$ and a G-orbit $\mathcal{O}$ in $\mathcal{E}$. Write $\omega_0 = \omega_0^\varepsilon$, regarding ω_0 as a fn of $\varepsilon \in [0,1]$.

<u>Case 1</u>: $\varepsilon = 0$. Put $\omega_0^0 := R(\tilde{\omega})$, Then $\iota_0 : \mathcal{O} \to \mathbb{R}$ takes its min at some pt $\theta \in \mathcal{O}$.

$\theta \leftrightarrow \lambda_{\theta;0}$ st $\theta = \omega_0^0(\lambda_{\theta;0})$, $A^0(\lambda_{\theta;0}) = -\lambda_{\theta;0} + f_0$

$(\mathrm{Hess}\,\iota_0)_\theta : H_\theta \times H_\theta \to \mathbb{R}$ is positive <u>semi-def</u>.

<u>Case 2</u>: $\varepsilon > 0$. Set $\omega_0^\varepsilon = (1-\varepsilon)\omega_0^0 + \varepsilon\theta = \omega_0^0(\varepsilon\lambda_{\theta;0}) \leftrightarrow$

Set $\lambda_{\theta;\varepsilon}$ st $\theta = \omega_0^\varepsilon(\lambda_{\theta;\varepsilon})$

$= \omega_0^0(\lambda_{\theta;0}) = \omega_0^0(\varepsilon\lambda_{\theta;0} + \lambda_{\theta;\varepsilon})$

$\Rightarrow \lambda_{\theta;\varepsilon} = (1-\varepsilon)\lambda_{\theta;0} + C_\varepsilon$, $\underbrace{\int \lambda_{\theta;\varepsilon}\varphi\,\theta^n}_{\iota(\omega_0^\varepsilon,\theta)} = \int (1-\varepsilon)\lambda_{\theta;0}\varphi\,\theta^n = 0$ if $\varphi \in H_\theta$

$\Rightarrow \theta$ is a critical pt for $\iota_\varepsilon : \mathcal{O} \to \mathbb{R}$

$$(\mathrm{Hess}\, \iota_\varepsilon)_\theta(\varphi, \varphi) = \int (1+\tfrac{1}{2}\Delta_\theta \lambda_{\theta;\varepsilon})\,\varphi^2\theta^n/V$$

$$= \int \Big(1+\tfrac{1}{2}\Delta_\theta\big((1-\varepsilon)\lambda_{\theta;0}+C_\varepsilon\big)\Big)\varphi^2\theta^n/V$$

$$(1+\tfrac{1}{2}(1-\varepsilon)\Delta_\theta(\lambda_{\theta;0}))\varphi^2\theta^n/V$$

$$= (1-\varepsilon)\int(1+\tfrac{1}{2}\Delta_\theta\lambda_{\theta;0})\varphi^2\theta^n/V + \varepsilon\int\varphi^2\theta^n/V$$

$$= (1-\varepsilon)(\mathrm{Hess}\,\iota_0)_\theta(\varphi,\varphi) + \varepsilon\int\varphi^2\theta^n/V > 0.$$

Mabuchi functional:

$\theta \mapsto \lambda_\theta$ extends to family

$$\Rightarrow M(\theta, \omega_0^\varepsilon(\varphi_{0;\varepsilon})) = \mu(\omega_0^\varepsilon(\varphi_{0;\varepsilon})) - \mu(\omega_0^\varepsilon(\varphi_1^{\lambda_{\theta;\varepsilon}})) \geq 0$$

$$\text{since} \quad \frac{d}{dt}\mu(t) = -(1-t)\frac{d}{dt}(I-J) \leq 0$$

$R(\omega_0^\varepsilon(\varphi_{0;\varepsilon})) = \omega_0^\varepsilon$ and $R(\tilde\omega) = \omega_0^0$

$\omega_0^\varepsilon \to \omega_0^0$ in $C^{0,\alpha}$ as $\varepsilon \to 0$ $\Rightarrow$ $\omega_0^\varepsilon(\varphi_{0;\varepsilon}) \to \tilde\omega$ in $C^{2,\alpha}$

$\Rightarrow M(\theta, \tilde\omega) \geq 0$, ie $\mu^+(\theta) \leq \mu^+(\tilde\omega)$, and μ^+ is const on $\mathcal{O}$

and $\mathcal{E}$ = critial pts of μ^+.