Numerical Characterization of the Kähler Cone of a Compact Kähler Current (Demailly - Paun).

Let X be a compact complex manifold (not nec. Kähler) and $\{\alpha\} \in H^{1,1}_{\partial\bar\partial}(X,\mathbb{R})$, and ω a pos. definite Hermitian (1,1) form

Def'n: $\{\alpha\}$ is NEF if, for every $\varepsilon > 0$ there is a rep.

$$\alpha_\varepsilon = \alpha + i\partial\bar\partial\varphi_\varepsilon \in \{\alpha\} \quad \text{s.t.} \quad \alpha_\varepsilon \geq -\varepsilon\omega.$$

- if X is Kähler, then $\{\alpha\}$ is NEF iff $\{\alpha\} \in \overline{K}$, where K is the Kähler cone.
- if $X \subseteq \mathbb{P}^N$ and $\{\alpha\}$ is the class of a Divisor, then $\{\alpha\}$ is NEF iff D is a NEF divisor.

Def'n: Recall that a current of pure bidegree (1,1) is

$$T = i\sum_{p,q} T_{\bar p q}\, dz^q \wedge d\bar z^p \quad \text{where } T_{\bar p q} \text{ is a distribution}$$

on X. Moreover, T is <u>positive</u> if

$$T \wedge iu_1 \wedge \bar u_1 \wedge \cdots \wedge iu_{n-1} \wedge \bar u_{n-1} \text{ is a positive}$$

measure $\forall\, u_i$ smooth (1,0) forms on X.

Recall that the de Rham differential acts on currents by integration by parts. ie (for a (1,1) form)

$$\int dT \wedge \psi = (-1) \int T \wedge d\psi$$

Def'n T is closed if $dT = 0$. A Kähler current on X is a closed positive current of bidegree (1,1) s.t. $T \geq \varepsilon\omega$ (ie $T - \varepsilon\omega$ is a pos. current) for some $\varepsilon > 0$

Def'n: If T is a closed, positive current of bidegree (1,1), Then we can write T locally as $T = i\partial\bar{\partial}\varphi$. for some PSH function φ, then the Lelong number is

$$\nu_T(a) := \liminf_{z \to a} \frac{\varphi(z)}{\log|z-a|}.$$

The Lelong number is a good (but crude) measure of the singularities of T.

Eg: $\varphi = \log|z|$. Then $\nu_\varphi(0) = 1$ so Lelong detects the singularity. But $\varphi = \log\log|z|$ has $\nu_\varphi(0) = 0$, so it misses the sing.

Regularization of (1,1) currents.

Let $T = \alpha + i\partial\bar{\partial}\psi$ be a closed (1,1)-current on X, α smooth ψ quasi-PSH. Assume $T \geq \gamma$, γ Real (1,1) form on X w/ real coefficients. Then $\exists\, T_k = \alpha + i\partial\bar{\partial}\psi_k$ of closed (1,1)-currents s.t.

(i) ψ_k is C^∞ on $X \setminus Z_k$, Z_k analytic set, and

$$Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_k \subseteq \cdots \subseteq X.$$

(ii) $T_k \geq \gamma - \delta_k \omega$ with $\lim_{k\to\infty} \downarrow \delta_k = 0$

(iii) (ψ_k) is non-increasing and $\lim \downarrow \psi_k = \psi$. Ie $T_k \to T$ weakly.

(iv) Near Z_k, the potential ψ_k satisfies

$$\psi_k = \lambda_k \log\Big(\sum_\ell |g_{k,\ell}|^2\Big) + O(1)$$

for $(g_{k,\ell})$ hol'c, $\lambda_k > 0$.

Main Point: if T is a Kähler current, then we can produce $\tilde{T}$ Kähler with analytic singularities. Then $\tilde{T}$ is smooth in a Nbhd of any point where $\nu_p(\tilde{T}) = 0$.

That is: The set $E_+(T) = \{ p \in X \mid \nu_T(p) > 0 \}$ is the "obstruction" to T being a genuine Kähler metric when T has analytic singularities.

Theorem (Siu)

For $c > 0$, T positive, closed $(1,1)$ current, the set $E_c(T) := \{ p \in X \mid \nu_T(p) \geq c \}$ is ~~analytic~~ an analytic subset of X.

Our goal for today:

Theorem (Demailly-Paun)

Let X be a compact Kähler mfd. Then the Kähler cone K of X is one of the connected components of

$$P = \left\{ \{\alpha\} \in H^{1,1}(X, \mathbb{R}) \;\middle|\; \int_Y \alpha^{\dim Y} > 0 \text{ for every irreducible analytic set } Y \subseteq X,\ \dim Y > 0 \right\}$$

Notice that $K \subseteq P$ is obvious, and K is open, so $K \subseteq P^{int}$. Now, if we can show that K is closed in P, the theorem will follow.

Let $\{\alpha\} \in \overline{\mathcal{K}} \cap \mathcal{P}$. Then $\{\alpha\}$ is NEF, and satisfies

$\int_Y \alpha^{\dim Y} > 0 \quad \forall\, Y \subseteq X$. We need to show that this implies $\{\alpha\}$ is Kähler.

Let's consider the case $\dim_{\mathbb{C}} X = 1$. Let ω be a Kähler metric on X, and $p \in X$. Consider Poisson's equation

$$\frac{i}{2\pi} \partial\bar\partial u = \omega .$$

This has a solution with a logarithmic singularity a p, and smooth outside p. Then: fix $\varepsilon < 1$, ~~and~~ so

$$T = \alpha + i\partial\bar\partial\varphi_\varepsilon + \frac{i}{2\pi}\partial\bar\partial u = \alpha + i\partial\bar\partial\varphi_\varepsilon + \omega > (1-\varepsilon)\omega$$

So T is a Kähler current w/ a log singularity at p. But this is easy to control since the function

$c|z-p|^2$ has positive Hessian in a Nbhd of p.

Gluing this in carefully (ie using the Regularized max) we have

$$\alpha + i\partial\bar\partial\left(\widetilde{\max}\left(\varphi_\varepsilon + \frac{u}{2\pi},\ c|z-p|^2\right)\right) \text{ is Kähler.}$$

Strategy of the Proof:

(1) Use Mass concentration in Calabi-Yau Theorem to produce Singular Kähler metrics. (i.e. Kähler currents) $\tilde{T} \geq \varepsilon\omega$

(2) Use Demailly Reg. to moderate Singularities.
$T \geq \frac{\varepsilon}{2}\omega$, T w/ analytic singularities. Then,
T is a Kähler metric outside of $\{p \in X \mid \nu_T(p) > 0\}$.

(3) Fix $c > 0$. Then $E_c(T) = \{p \in X \mid \nu_T(p) \geq c\}$ is an analytic subset, $Y = \dim(E_c(T)) \leq \dim X - 1$.
Also $\{\alpha\}|_Y$ is NEF, $\int_Z \alpha^{\dim Z} > 0$ $\forall Z \subseteq Y$ $\dim Z > 0$.

Now use induction to produce a Kähler metric on Y in the class $\{\alpha\}|_Y$. Now extend to a small Nbhd on Y, and glue to T. Then T' is Kähler on $T \setminus E_c(T) \cup Y$. Continue.

This is exactly what we did in dimension 1, except step (1) Requires more technology.

<u>Step 1</u>: Mass concentration. This is a generalization of the method ~~...~~ used in Paun's paper earlier this spring.

~~Lem~~ <u>Proposition 2.6</u>: Let (X,ω) be a cpct, Kähler n-fold, $\{\alpha\} \in H^{1,1}(X,\mathbb{R})$ NEF, with $\alpha^n > 0$. Then, $\forall$ p-codimensional analytic subsets $Y \subseteq X$, $\exists$ a closed, positive current $\Theta \in \{\alpha\}^p$ of bidegree (p,p) such that $\Theta \geq \delta[Y]$ for some $\delta > 0$.

Recall in <u>Paun</u> we proved this when $X \subseteq \mathbb{P}^N$ and Y was an ample divisor. We need the following:

<u>Lemma</u> 2.1 (X,ω) Kähler, $Y \subseteq X$ analytic subset. Then $\exists$ quasi PSH potentials ψ, $(\psi_\varepsilon)_{\varepsilon\in(0,1]}$ on X s.t.

(i) ψ is C^∞ on $X \setminus Y$, $i\partial\bar\partial\psi \geq -A\omega$ for $A>0$, and ψ has log. singularities along Y.

(ii) $\psi = \lim_{\varepsilon\to 0} \searrow \psi_\varepsilon$, $\psi_\varepsilon \in C^\infty$, and $i\partial\bar\partial\psi_\varepsilon \geq -A\omega$.

(iii) Set $\omega_\varepsilon = \omega + \frac{1}{2A} i\partial\bar\partial\psi_\varepsilon \geq \frac{1}{2}\omega$. Fix $x_0 \in Y$, U a Nbhd of x_0, $V_\varepsilon = \{z \in X : \psi(z) < \log\varepsilon\}$. Then

$$\int_{U\cap V_\varepsilon} \omega_\varepsilon^n \geq \delta(U) > 0.$$

(iv) $\forall\, p \geq 0$ ω_ε^p is bdd in Mass i.e. $\int_X \omega_\varepsilon^p \wedge \omega \leq C$.

Also, if $Y' \subseteq Y$ is an irr. component of codimension p, then

$$\int_{U \cap V_\varepsilon} \omega_\varepsilon^p \wedge \omega^{n-p} \geq \delta_p(U) > 0.$$

For any Nbhd U of a regular pt. $x_0 \in Y'$. In particular, any weak limit Θ of ω_ε^p as $\varepsilon \to 0$ satisfies $\Theta \geq \delta'[Y']$ for some $\delta' > 0$.

<u>Idea</u>: Take generators for the ideal sheaf $\mathcal{I}_Y$ and mimic the process of Paun.

pf of Proposition 2.6. (Sketch).

Given Y, construct ω_ε as above. Since $\alpha + \varepsilon\omega$ is a Kähler class we can solve

$$\alpha_\varepsilon^n := \left(\alpha + \varepsilon\omega + i\partial\bar\partial\varphi_\varepsilon\right)^n = c_\varepsilon\, \omega_\varepsilon^n$$

where $c_\varepsilon = \int_X (\alpha+\varepsilon\omega)^n \Big/ \int \omega_\varepsilon^n = \dfrac{\int_X (\alpha+\varepsilon\omega)^n}{\int_X \omega^n} \geq \dfrac{\int \alpha^n}{\int \omega^n} \geq c > 0$

for $c > 0$. This is <u>crucial</u>!

Now take Θ to be a weak limit of α_ε^p as $\varepsilon \to 0$.
(Checking everything is non-trivial!! See D.P.

∎

Theorem 2.12: Let (X,ω) be Kähler, $\{\alpha\} \in H^{1,1}(X,\mathbb{R})$ NEF s.t. $\int_X \alpha^n > 0$. Then $\{\alpha\}$ contains a Kähler current.

Pf (Sketch) Apply Prop 2.6 to $\tilde{Y} = \Delta \subseteq X \times X = \tilde{X}$. where we put $\tilde{\omega} = \pi_1^*\omega + \pi_2^*\omega$ the product metric.

~~Note~~ and $\tilde{\alpha} = \pi_1^*\alpha + \pi_2^*\alpha$ which is clearly NEF on $\tilde{X}$. Note also

$$\int_{X\times X} \tilde{\alpha}^{2n} = \int_{X\times X} (\pi_1^*\alpha + \pi_2^*\alpha)^{2n} = \binom{2n}{2}\left(\int_X \alpha^n\right)^2 > 0.$$

Since $(\pi_1^*\alpha)^p \wedge (\pi_2^*\alpha)^q = 0$ if $p \neq q$. (ie $p=q=n$)

Then we get Θ an (n,n) current in $\{\tilde{\alpha}^n\}$. s.t. $\Theta \geq \varepsilon[\Delta]$ for some $\varepsilon > 0$. define $T = c\,\pi_{1*}(\Theta \wedge \pi_2^*\omega)$ is a $(1,1)$ current with

$$T \geq c\,\varepsilon\,\pi_{1*}([\Delta] \wedge \pi_2^*\omega) = c\varepsilon\omega.$$

~~By~~ choosing c appropriately, $T \in \{\alpha\}$. ∎