"On Regularization of Plurisubharmonic Functions on Manifolds"

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Let M be a complex manifold.

Def'n A function $\varphi : M \to \mathbb{R} \cup \{-\infty\}$ is quasi-plurisubharmonic if it can be locally written as a sum of a smooth function and a psh fn.

Def'n For a continuous $(1,1)$-form γ on M, $PSH(M,\gamma)$ will denote the class of quasi-psh fns φ on M s.t.

$$dd^c\varphi + \gamma \geq 0. \qquad (\gamma\text{-psh fns})$$

Note. if $\gamma \geq 0$, constants $\in PSH(M,\gamma)$

- $PSH(M,\gamma)$ is closed under operations of maximum and regularized maximum.

See Demailly Lemmas I.5.17–5.18

Def'n The Lelong number of φ at $a \in M$ is

$$\nu_\varphi(a) = \liminf_{z \to a} \frac{\varphi(z)}{\log|z-a|}$$

or

$$= \lim_{r \to 0} \left(\frac{\sup_{|z-a|=r} \varphi(z)}{\log r} \right)$$

Theorem 1

Let ω be a positive continuous $(1,1)$-form on a compact mfld M.

For every $\varphi \in PSH(M,\omega)$, $\exists$ a sequence $\varphi_j \in PSH(M,\omega) \cap C^\infty(M)$

Does not need to be a Hodge form like in G-Z 2

Such that φ_j decreases to φ.

Theorem 2 Let M be a cmplx mfld w/ a pos. hermitian form ω

Asssume that γ is a continuous $(1,1)$ form on M, and let $\varphi \in PSH(M,\gamma)$ be such that the Lelong number $\nu_\varphi(z) = 0$ $\forall$ $z \in M$. (true if φ locally bdd)

Then for every open $M' \subset M$ we can find $\varepsilon_j \searrow 0$ and $\varphi_j \in PSH(M', \gamma + \varepsilon_j \omega) \cap C^\infty(M')$ decreasing to φ in M'

Theorem 3 (comparison principle for the complex Monge-Ampère operator)

Let (M,ω) be a compact kähler mfld.

For $\varphi, \psi \in PSH(M,\omega) \cap L^\infty(M)$,

$$\int_{\{\varphi<\psi\}} (dd^c\psi + \omega)^n \leq \int_{\{\varphi<\psi\}} (dd^c\varphi + \omega)^n$$

Proof Thm 1

- WLOG, we may assume $\varphi \leq -1$.
- Since ω is positive, constants $\in PSH(M, \omega)$, so we have a sequence

$\max\{\varphi, -j\} \in PSH(M, \omega) \cap L^\infty(M)$

which decreases to φ.

Then we may assume φ is bounded.

Then $\nu_\varphi(z) = 0 \ \forall z \in M \Rightarrow$ Apply Thm 2 with

$\gamma = \omega$, $M' = M$ (allowed since M compact).

Then $\exists \ \varepsilon_j \searrow 0$ and $\psi_j \in PSH(M, \lambda_j \omega) \cap C^\infty(M)$, with $\lambda_j := 1 + \varepsilon_j$,

such that $\psi_j \searrow \varphi$ in M (can assume $\psi_j < 0$)

Let $\varphi_j := \dfrac{\psi_j}{\lambda_j}$. Since $\psi_j < 0$, $\psi_j \searrow \varphi$, and $\lambda_j \searrow 1$,

we have $\varphi_j \searrow \varphi$.

Since $\psi_j \in PSH(M, \lambda_j \omega) \cap C^\infty(M)$, $dd^c \psi_j + \lambda_j \omega \geq 0$

$$\Rightarrow dd^c \frac{\psi_j}{\lambda_j} + \omega \geq 0$$

$$\Rightarrow dd^c \varphi_j + \omega \geq 0$$

$$\Rightarrow \varphi_j \in PSH(M, \omega) \cap C^\infty(M)$$

and $\varphi_j \searrow \varphi$ $\square$

Pf of Thm 2

In the flat case, use the standard smooth regularization by convolution:

If $\rho(z) = \hat{\rho}(|z|) \in C_0^\infty(\mathbb{C}^n)$ is s.t.

(i) $\hat{\rho} \geq 0$

(ii) $\hat{\rho}(r) = 0$ for $r \geq 1$

(iii) $\int_{\mathbb{C}^n} \rho \, dx = 1$

(iv) $\rho_\delta(z) := \delta^{-2n} \rho\left(\frac{z}{\delta}\right)$ for $\delta > 0$

Then set

$$u_\delta(z) = (u * \rho_\delta)(z) = \int u(z - \delta w) \rho(w) \, d\lambda(w).$$

when u is psh $u_\delta \searrow u$ as $\delta \searrow 0$.

when u is continuous $u_\delta \to u$ locally uniformly

Lemma 4 Assume $U, V \underset{\text{open}}{\subseteq} \mathbb{C}^n$. Let $F: U \to V$ be biholo'.

Let $u \in PSH(U)$ s.t. $\nu_u(z) = 0 \ \forall z \in U$

Define $u_\delta^F := (u \circ F^{-1})_\delta \circ F$

Then $u_\delta - u_\delta^F \xrightarrow{\text{loc unif}} 0$ as $\delta \to 0$.

Fix $\varepsilon > 0$.

$\exists$ finite number of charts $V_\alpha \subset U_\alpha$ s.t. $\{V_\alpha\}$ covers M and

$\bigstar$ $0 \leq dd^c f_\alpha - \gamma \leq \varepsilon \omega$ in U_α

for some smooth functions f_α in U_α

Then $u_\alpha := \varphi + f_\alpha$ is plurisubharmonic in U_α.

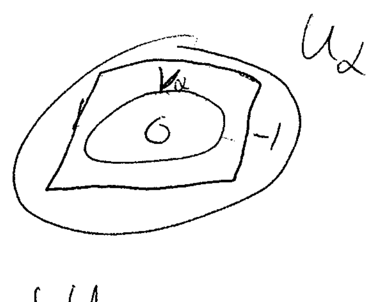
Pf Thm 2 (con't)

let $u^F_{\alpha,\delta}$ denote the regularization of u_α in U_β

By Lemma 4,

$$* \quad u_{\alpha,\delta} - u_{\beta,\delta} = u_{\alpha,\delta} - u^F_{\alpha,\delta} + (u_\alpha - u_\beta)^F_\delta \longrightarrow f_\alpha - f_\beta$$

locally unif in $U_\alpha \cap U_\beta$ as $\delta \to 0$.



Let η_α be smooth in U_α s.t.

$\eta_\alpha = 0 \quad$ in V_α

$\eta_\alpha = -1 \quad$ away from a compact subset of U_α

Then $dd^c \eta_\alpha \geq -C\omega$ for some constant C dependent on ε

Let $$\varphi_\delta := \max_\alpha \left(u_{\alpha,\delta} - f_\alpha + \frac{\varepsilon \eta_\alpha}{C} \right)$$

by $*$ if δ is sufficiently small, the values on $\{\eta_\alpha = -1\}$ do not effect the maximum, then φ_δ is continuous.

(If consider regularized maximum instead of maximum in above defin we get φ_δ smooth)

Using ✡, we get $\varphi_\delta \in PSH(M', \gamma + 2\varepsilon\omega)$,

Also, $\varphi_\delta \searrow \varphi$ as $\delta \to 0$ $\square$

Pf Lemma 4

Let $\hat{u}_\delta(z) := \max_{\overline{B(z,\delta)}} u.$

If u is psh $\Rightarrow$ $\hat{u}_\delta(z)$ is log convex in δ ~~$\Rightarrow \hat{u}_\delta$~~

$\Rightarrow$ $\hat{u}_\delta$ is continuous, psh in $U_\delta := \{z \in U;\ \overline{B(z,\delta)} \subset U\}$, and decreases to u as $\delta \searrow 0$

By log convexity, for $a \geq 1$, $r > 0$ fixed and $\delta \ll 1$

$$0 \leq \hat{u}_{a\delta} - \hat{u}_\delta \leq \frac{\log a}{\log\left(\frac{r}{\delta}\right)}\left(\hat{u}_r - \hat{u}_\delta\right)$$

Then if the Lelong #'s vanish, $\forall a > 0$

$$(\hat{*}) \qquad \hat{u}_{a\delta} - \hat{u}_\delta \to 0 \quad \text{locally uniformly as } \delta \to 0.$$

Also, let $\hat{u}_\delta^F := \widehat{(u \circ F^{-1})}_\delta \circ F = \max_{F^{-1}(\overline{B(F(z),\delta)})} u$

For ~~A~~ a fixed $K \subset U$, we can find $A > 1$ s.t. for $z \in K$ and $\delta \ll 1$

$$\overline{B}(F(z),\delta) \subset F(\overline{B}(z, A\delta)), \quad F(\overline{B}(z,\delta)) \subset \overline{B}(F(z), A\delta)$$

Then on K, $\hat{u}_\delta^F \leq \hat{u}_{A\delta}$, $\hat{u}_\delta \leq \hat{u}^F_{A\delta}$

$\overset{(\hat{*})}{\Rightarrow}$ $\hat{u}_\delta^F - \hat{u}_\delta \to 0$ locally uniformly in U as $\delta \to 0$

Then Lemma 5 For $u \in PSH(U)$, $V_u(z) = 0$ $\forall z \in U$.

$\hat{u}_\delta - u_\delta \to 0$ locally unif in U as $\delta \to 0$