Calabi - Yau Thm

M will be compact w/ fixed Kähler metric $\omega \leftrightarrow g$.

C^0 est

Theorem 1: Let f be positive smooth and let φ be ω-strictly psh. Assume

$$(\omega + i\partial\bar\partial\varphi)^n = f\omega^n.$$

Then have

$$\operatorname{osc}_M(\varphi) \leq C(M, g, \|f\|_\infty)$$

Note: Fix $\max_M \varphi = 0$ for simplicity. So we need to know how far φ can fall down.

Confusing notation! Since $g_{i\bar j}$ is Kähler it is locally given as $\frac{\partial^2 g}{\partial z_j \partial \bar z_k}$ for some potential function g. Up to constants then, we need to study local functions of the form

$$u := g + \varphi,$$

which will by necessity be strictly psh. So study how far this falls instead. There is freedom in our choice of g. We can add any ~~pluriharmonic~~ function to it without changing $g_{i\bar j}$.

<u>Compactness</u> $\Rightarrow \exists p \in M$ where $\varphi(p)$ is minimal.

Look at g at p. First diagonalize g at p. So

$$H_c(g) = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix},$$

where all a_i are positive. Now let's make g a little nicer. First if $k \neq j$,

$$g_{\bar{k}j} = 0 \Rightarrow \left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)\left(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\right) g = 0$$

$$\Rightarrow \left\{\left(\frac{\partial^2}{\partial x_j \partial x_k} + \frac{\partial^2}{\partial y_j \partial y_k}\right) + i\left(\frac{\partial^2}{\partial x_j \partial y_k} - \frac{\partial^2}{\partial y_j \partial x_k}\right)\right\} g = 0$$

$$\begin{cases} \left(\dfrac{\partial^2}{\partial x_j \partial x_k} + \dfrac{\partial^2}{\partial y_j \partial y_k}\right) g = 0 \\[2ex] \dfrac{\partial^2 g}{\partial x_j \partial x_k} = -\dfrac{\partial^2 g}{\partial y_j \partial y_k} \end{cases}$$

So now change g by adding pieces so <u>this</u> remains true, but now

$$\frac{\partial^2 g}{\partial x_j \partial x_k} = 0 = \frac{\partial^2 g}{\partial y_j \partial y_k}$$

Do the same with the imaginary guys and get that mixed <u>unequal</u> <u>real</u> partials vanish at p.

Now have the case $i=j$. Then have that

$$\frac{\partial^2 g}{\partial z_j \partial \bar{z}_j} = \Delta(g).$$

So now add functions so that

$$\frac{\partial^2 g}{\partial x_j^2} = \frac{\partial^2 g}{\partial y_j^2} = \frac{a_j}{2}$$

<u>Conclusion</u>: g is now strictly convex at p and has lower position bounds on the eigenvalues. NONE OF THIS EVER TOUCHED φ!

So now we could cover M by opens with g rigged in this way so that there is always some fixed lower bound on the Hessian.

„More Convex" $\Longleftrightarrow$ „Bigger metric"

Now want to use this fixed information on g to control φ.

This will be used more later...

<u>Theorem (Cheng-Yau)</u>: Let $\Omega \subset \mathbb{C}^n$ be bounded domai
Assume $u \in \mathcal{C}(\bar\Omega)$ is psh and $\mathcal{C}^2$ in Ω with $u = 0$ on $\partial\Omega$. Then

$$\| u \|_{L^\infty(\Omega)} \le C(n, \operatorname{diam}\Omega) \| f \|^{1/n}_{L^2(\Omega)},$$

when $f = \det(u_{\bar i j})$.

Proof translates ABP to complex setting.

<u>Lemma (ABP)</u>: λ_{2n} is vol unit ball

$$\| u \|_{L^\infty(\Omega)} \le \frac{\operatorname{diam}(\Omega)}{\lambda_{2n}^{1/2n}} \left(\int_\Gamma \det D^2 u \right)^{1/2n},$$

when Γ is „contact set."

$$\Gamma := \{ x \in \Omega \mid u(x) + \langle Du(x), y - x \rangle \le u(y), \forall y \in \Omega \}$$

$$\subset \{ D^2 u \ge 0 \}$$

Now need to show: $D^2 u \ge 0 \Rightarrow \det D^2 u \le c_n (\det (u_{\bar i j}$
Assume $u_{\bar i j}$ <u>diagonal</u>:

$$\det(u_{\bar i j}) = 4^{-n} (u_{x_1 x_1} + u_{y_1 y_1}) \cdots (u_{x_n x_n} + u_{y_n y_n})$$

true for <u>non-neg</u> symmetric

$$\ge 2^{-n} \sqrt{u_{x_1 x_1} u_{y_1 y_1} \cdots u_{x_n x_n} u_{y_n y_n}}$$

$$\ge \sqrt{\det D^2 u / c_n}$$

<u>qed</u>

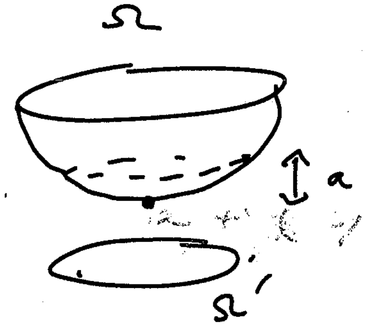
Proposition: Let Ω be bounded in $\mathbb{C}^n$, u psh and negative. Assume $a > 0$ and that the set $\{u < \inf_\Omega u + a\}$ is non-empty and relatively cpt.

$$\Rightarrow \|u\|_{L^\infty(\Omega)} \le a + (C/a)^{2n} \|u\|_{L^1(\Omega)} \|f\|^2_{L^\infty(\Omega)},$$

where C from last Thm and $f = \det(u_{i\bar{j}})$.

Proof: • $t := \inf_\Omega u + a$ • $\Omega' = \{v < 0\}$
• $v := u - t$

So $v \le 0 \Rightarrow u \le t \Rightarrow$ „u is less than a above m

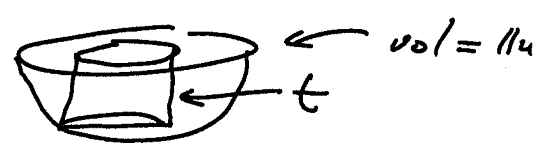


want from $L^2 \to L^\infty$
by positivity / integrati
and putting vol there

Last theorem says:

$$a = \|v\|_{L^\infty(\Omega')} \le C(\mathrm{vol}(\Omega'))^{1/2n} \|f\|^{1/n}_{L^\infty(\Omega')}$$

But have also:



$$\mathrm{vol}(\Omega') \le \frac{\|u\|_{L^1(\Omega)}}{|t|} = \frac{\|u\|_{L^1(\Omega)}}{\|u\|_{L^\infty(\Omega)} - a}$$

Hence have:

$$a^{2n} \leq C^{2n} \left(Vol(\Omega')\right) \|f\|^2_{L^\infty(\Omega)}$$

$$\leq C^{2n} \frac{\|u\|_{L^1(\Omega)}}{\|u\|_{L^\infty(\Omega)} - a} \|f\|^2_{L^\infty(\Omega)}$$

$$\Rightarrow \|u\|_{L^\infty(\Omega)} \leq a + \left(\frac{C}{a}\right)^{2n} \|u\|_{L^1(\Omega)} \|f\|^2_{L^\infty(\Omega)}$$

$\boxed{\text{qed}}$

So now to finish the theorem would like to argue as follows:

Pick $p \in M$ where φ takes min. Choose an open set around it and a nice potential for g so that g grows nicely. Hence $g+\varphi$ will grow up at least to $2a$ for some a dp only on g.

Then applying last theorem to $u = g + \varphi$ we get and L^∞ bound on φ with the price of $\|u\|_{L^1}$ floating around. Since g is understood, we would be done if we could establish and L^1 bound on φ which is universal.

<u>Lemma</u>: Bound holds for any φ that is g-psh. Follows from Green's or some other magic of Blocki.

C^2-est

<u>Theorem</u>: Let $\varphi \in \mathcal{C}^4(M)$ satisfy $\omega + i\partial\bar\partial\varphi > 0$ and $(\omega + i\partial\bar\partial\varphi)^n = f\omega^n$.

$$\Rightarrow \sup_M |\Delta\varphi| \leq C(M, g, \|f^{1/n-1}\|_{1,1}).$$

To prove can now assume $-C_1 \leq \varphi \leq 0$. Since φ is admissable, $\Delta\varphi \geq -n$ so only need the upper bound.

Really just need to estimate the second derivative of u. So now choose a direction. So we need to estimate the following quantity:

$$\eta = \max_{|\xi|=1} \frac{u_{\bar\xi\xi}}{g_{\bar\xi\xi}} = \max_{|\xi|=1} \frac{u_{\bar\xi\xi}}{g_{\bar\xi\xi}}$$

η is ind of holomorphich coordinate change and thus a continuous, <u>positive</u> globally defined function on M. Define a quantity: <u>Note</u>: $\varphi \leq 0$!

$$\alpha := \log\eta - A\varphi, \quad A > 0$$

and specify A later. Pick $y \in M$ where α is max. Assume $(u_{\bar i j})$ diagonal at y, $u_{\bar 1 1} \geq \cdots \geq u_{\bar n n}$ at y. Fix $\xi \in \mathbb{C}^n$, $|\xi| = 1$, such that

$$\eta = \frac{u_{\bar\xi\xi}}{g_{\bar\xi\xi}} \quad \text{at } y.$$

Now define the function $\tilde{\alpha}$ by

$$\tilde{\alpha} := \log \frac{u_{\bar{\xi}\xi}}{g_{\bar{\xi}\xi}} - A\varphi$$

So then $\tilde{\alpha}(y) = \alpha$, $\tilde{\alpha} \leq \alpha$, $\tilde{\alpha}$ smooth!

<u>Lemma</u>: Let u be C^4 psh with $F := \det(u_{\bar{i}j}) > 0$. Then for any ξ,

$$u^{j\bar{k}} (\log u_{\bar{\xi}\xi})_{\bar{k}j} \geq \frac{(\log F)_{\bar{\xi}\xi}}{u_{\bar{\xi}\xi}}$$

<u>Proof</u>: Computation.

Need to control $\tilde{\alpha}(y)$.

$$\tilde{\alpha} = \log \frac{u_{\bar{y}y}}{g_{\bar{y}y}} - A\varphi$$

@ y:

$$\begin{aligned} 0 &\geq u^{j\bar{k}} \tilde{\alpha}_{\bar{k}j} \\ &= \underbrace{u^{j\bar{k}} (\log u_{\bar{y}y})_{\bar{k}j}}_{①} - \underbrace{u^{j\bar{k}} (\log g_{\bar{y}y})_{\bar{k}j}}_{③} - \underbrace{A u^{j\bar{k}} \varphi_{\bar{k}j}}_{②} \end{aligned}$$

① $u^{j\bar{k}} (\log u_{\bar{y}y})_{\bar{k}j} \geq \dfrac{(\log F)_{\bar{y}y}}{u_{\bar{y}y}}$, $\quad F = \det(u_{\bar{b}a}) > 0$

② $u^{j\bar{k}} \varphi_{\bar{k}j} = u^{j\bar{k}} u_{\bar{k}j} - u^{j\bar{k}} g_{\bar{k}j} = n - u^{j\bar{k}} g_{\bar{k}j}$

③ $$\begin{aligned} u^{j\bar{k}} (\log g_{\bar{y}y})_{\bar{k}j} &= u^{j\bar{k}} \left(g_{\bar{y}y}^{-1} g_{\bar{y}y\bar{k}} \right)_j \\ &= u^{j\bar{k}} g_{\bar{y}y}^{-2} \left(g_{\bar{y}y} g_{\bar{y}y\bar{k}j} - \cancelto{0}{g_{\bar{y}y\bar{k}}} g_{\bar{y}yj} \right) \end{aligned}$$

$F = f \det(g_{\bar{k}j})$

Combining, we get:

$$u^{j\bar{k}} \tilde{\alpha}_{\bar{k}j} \geq \overbrace{\frac{(\log f)_{\bar{y}y}}{u_{\bar{y}y}} + \frac{(\log \det(g_{\bar{b}a}))_{\bar{y}y}}{u_{\bar{y}y}}}^{①}$$

$$+ A u^{j\bar{k}} g_{\bar{k}j} - An \qquad ②$$

not there due to mistake!

$$- u^{j\bar{k}} (\log g_{\bar{y}y})_{\bar{k}j} \qquad ③$$

Can ignore ③ by absorbing it in A!

So left w/ inequality:

$$0 \geq \frac{(\log f)_{\bar{y}y}}{u_{\bar{y}y}} + \frac{(\log \det(g_{\bar{i}k}))_{\bar{y}y}}{u_{\bar{y}y}} + A u^{j\bar{k}} g_{\bar{k}j} - nA$$

Use following: $\|\sqrt{h}\|_{0,1} \leq C_M(1 + \|h\|_{1,1})$, $h \in C^2(M)$, $h \geq 0$.

Denote $\tilde{f} = f^{1/(n-1)}$. Hence

$$\frac{(\log f)_{\bar{y}y}}{u_{\bar{y}y}} = \frac{n-1}{u_{\bar{y}y}}\left(\frac{\tilde{f}_{\bar{y}y}}{\tilde{f}} - \frac{|\tilde{f}_y|^2}{\tilde{f}^2}\right) \geq -\frac{C_4}{u_{11}\tilde{f}}$$

$$f'' - \frac{(f')^2}{f} \overset{?}{\geq} -c$$

$$f f'' - (f')^2 \overset{?}{\geq} -cf$$