Kołodziej's Stability Theorem (from book) 3/26

(M, ω) cpt Kähler. A [illegible] fcn φ is (L^∞) called ω-psh if $\omega + i\partial\bar\partial\varphi \geq 0$ as a (closed) current (differential form w/ measure coeffs, to be wedged & integrated over M.) [illegible]

Def: For $E \subset M$ Borel, define the capacity

$$\mathrm{cap}_\omega(E) = \sup\left\{ \int_E \omega_\varphi^n \;\middle|\; \varphi \in PSH(\omega),\ 0 \leq \varphi \leq 1 \right\}.$$

The M-A measure ω_φ^n is defined inductively by [illegible], which [illegible] (closed) form ([illegible])

Kol. Theorem 1.11 ("convergence thm"): If a unif. bdd $\leq C$ sequence $\{\varphi_j\} \subset PSH(\omega)$ converges to φ w.r.t. capacity (i.e. $\forall t > 0$ $\lim_{j\to\infty} \mathrm{cap}(\{x \in M : |\varphi_j - \varphi| > t\}) = 0$), [where the set $= E_t$]

then $\omega_{\varphi_j}^n \to \omega_\varphi^n$

Proof: $\omega_{\varphi_j}^n - \omega_\varphi^n = (\omega_{\varphi_j} - \omega_\varphi) \wedge \left(\sum \omega_{\varphi_j}^k \wedge \omega_\varphi^{n-1-k}\right)$ [$= T$, closed, pos, by induction.]

$= dd^c(\varphi_j - \varphi) \wedge T$

For $\Phi \in$ [illegible] smooth test fcn

$$\int_E \Phi\, dd^c(\varphi_j - \varphi) \wedge T \leq \int_{E_t} |\varphi_j - \varphi|\, dd^c\Phi \wedge T + t\, \| dd^c\Phi \wedge T \|_E$$

$\leq \|dd^c \phi\| C^{?} \left(C \cdot \mathrm{cap}_\omega E_t + t\, \mathrm{cap}_\omega E \right)$

↳ constant needed to bound $\|T\|$ by cap (maybe just C^n?)

[illegible]

(? one way or another need to compare capacities.)

For weak convergence, ϕ is fixed. Let $t \to 0$, done.

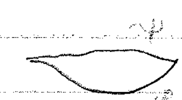
(global version)

<u>Prop 1.12:</u> a decreasing sequence $\varphi_j \downarrow \varphi \in PSH(\omega)$ converges w.r.t. capacity $\therefore \omega_{\varphi_j} \to \omega_\varphi$.

Nontrivial. Needed for the following main tool:

<u>Comparison Principle:</u>

$$\int_{\{\varphi < \psi\}} \omega_\varphi^n \leq \int_{\{\varphi < \psi\}} \omega_\psi^n$$

<u>Proof:</u> smooth case: Stokes (φ more convex than ψ )

non-smooth: [illegible] convergence theorems.

~~[illegible] lip service to Ch. 1 / prelims Ch. 6~~

Last fact: uniform L^1 bound

$\sup_M \varphi = 0 \Rightarrow \int_M |\varphi| \omega^n < B$

((Green's fn, or sub-mvp on balls)).

End of my lip service to Ch. 1 / prelims Ch. 6 / previous talks.

Now: Kolodziej's notation

Def: A str. increasing fcn $h : \mathbb{R}_+ \to (1, \infty)$ is called "admissible" if

$$\int_1^\infty (x\, h^{1/n}(x))^{-1}\, dx < \infty$$

and if for some $a > 1$, $b \geq 1$ and $x_0 > 0$ we have $h(ax) \leq b\, h(x)$ for $x > x_0$.

i.e. any increasing polynomial on $\mathbb{R}_+$: $h(ax) \leq (a^d + \varepsilon) h(x)$ for $x > x_0$.

THINK: $h(x) = x^n$.

Consider the family of fcns

Def: $\mathcal{F}(A, h) = \{ f \in L^1(M) : f \geq 0, \int_M f \omega^n = 1,$

$$\int_E f \omega^n \leq F(\mathrm{Cap}_\omega(E)) \quad \forall E \subset M \text{ Borel} \},$$

where $F(x) = \dfrac{Ax}{h(x^{-1/n})}$, $A > 0$ and h admissible.

Think: $F(x) = A x^{1+\alpha} = A x^2$

Step 1 of Valentino's proof: needed

$$\int_E \omega_\varphi^n = \int_E \mu = \int_E f \omega^n \leq A \cdot (\mathrm{Cap}_\omega E)^{1+\alpha}$$

could be any $\nu = \frac{1}{n}$ degh.

Used for iteration (RHS)

To be precise,

<u>Lemma 1</u>: For A big enough, $1 \in \mathcal{F}(A,h)$,

i.e. $\int_E \omega^n \leq A\, Cap_\omega(E)^{1+\alpha}$. (holds for ω non-smooth?, n.nef.)

Proof used · Siciak Extr. fcn (to max out capacity)
· Alexander–Taylor (to compare w/ V_E ... (E))
· alpha invariant.

Note: "converse" is easier: control cap of sublevel sets by volume: $Cap_\omega(\{\varphi < -s-1\}) \leq \int_{\{\varphi < -s\}} \omega_\varphi^n$.

<u>Proof</u>: let $0 \leq u \leq 1$ psh test fcn:

$$\{\varphi < -s-1\} \subset \{\varphi < u-s-1\} \subset \{\varphi < -s\}$$

$$\Rightarrow \int_{\{\varphi<-s-1\}} (\omega + dd^c u)^n \leq \int_{\{\varphi < u-s-1\}} (\omega + dd^c(u-s-1))^n \overset{C.P.}{\leq} \int_{\{\varphi < u-s-1\}} (\omega + dd^c\varphi)^n$$

$$\leq \int_{\{\varphi < -s\}} \omega_\varphi^n.$$

Today we're interested not in sub-level sets, but in comparing solutions φ, ψ

<u>Lemma 2</u>: assume $0 \leq \psi \leq C$, $t < 1$, $0 < t \leq \frac{s+D-S}{C+1}$.

Then $t^n\, Cap_\omega(\{\varphi - s < \psi\}) \leq \int_{\{\varphi - s - t(1+C) < \psi\}} \omega_\varphi^n$

Proof: let $0 \le \rho \le 1$ be a test fcn and

$$V(s) := \{\varphi - s - tC < t\rho + (1-t)\psi\}.$$

note $\varphi - s < \psi \Rightarrow \{\varphi - s - t \cdot C < \psi - t\psi + t\rho\}$

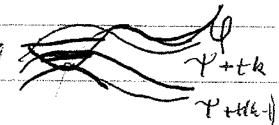
$$\Rightarrow \{\varphi - s - tC - t < \psi\}.$$

$$t^n \int_{\{\varphi - s < \psi\}} \omega_\rho^n \le \int_{\{\varphi - s - t \cdot C \le t\rho + (1-t)\psi\}} (t\omega_\rho + (1-t)\omega_\psi)^n$$

$$\overset{\text{c.p.}}{\le} \int_{\text{''}} \omega_\varphi^n \le \int_{\{\varphi - s - t - tC < \psi\}} \omega_\varphi^n \qquad \square$$

Put together lemmas 1 & 2:

$$t^n \operatorname{cap}_\omega\{\varphi - s < \psi\} \overset{L2}{\le} \int_{\{\varphi - s - t - tC < \psi\}} d\mu \overset{L1}{\le} A \cdot \left(\operatorname{cap}_\omega\{\varphi - s - t(1+C) < \psi\}\right)^{1+\alpha}$$

We iterated this to get L^∞. Says that capacity has to drop off fast w/ s, once it gets small 

Kolodziej: same iteration shows that capacity above a nonempty overlap has to pick up fast (else contradiction.)

Many Vergina:

Main Lemma: Let φ, ψ be ω-psh fcns with $0 \le \psi \le C$. Assume $\{\varphi - S < \psi\} \neq \phi$.

Suppose $\int_K \omega_\varphi^n \le \frac{A \operatorname{cap}_\omega K}{h((\operatorname{cap}_\omega K)^{-1/n})}$, $A > 0$, $\forall K$ cpt.

(i.e. $\omega_\varphi^n = f\omega^n$ with $f \in \mathcal{F}(A,h)$, h admissible).

("Step 1")

Then there exists a cts increasing fcn $K_{A,h}(s)$ s.t. for all $D < 1$ we have

$$D \le K_{A,h}\left(\operatorname{cap}_\omega\{\varphi - S - D < \psi\}\right).$$

(lower bound on capacity.)

Proof: Let

$$K_{A,h}(s) = c(n) A^{1/n}(1+C)\left[\int_{s^{-1/n}}^{\infty} x^{-1} h^{-1/n}(x)\,dx + h^{-1/n}(s^{-1/n})\right]$$

□

In case $h(x) = x^n \Rightarrow \alpha = 1$, get

$$K_{A,h}(t) = \text{const}\cdot(At)^{1/n}.$$

Proof does iteration, quantitatively estimates # of steps until passes D.

Theorem:

Existence: If $1 \in \mathcal{F}(A,h)$, then for any $f \in \mathcal{F}(A,h)$ $\exists$ its solution φ, with uniform L^∞ bound $\operatorname{osc}\varphi \le a(A,h)$.

(already discussed.)

"Proof": Take smooth $f_j \to f$, have φ_j from Yau's theorem.

"$1 \in \mathcal{F}$" needed to conveniently have $\|f_j\|_\infty < N = N \cdot 1$
$\Rightarrow f_j \in \mathcal{F}(NA,h)$
$\Rightarrow \varphi_j$ unif. bdd.

Need super-good convergence lemmas about M-A masses, so $(\limsup \varphi_j)^* =: \varphi$ solves $\omega_\varphi^n = f\omega^n$.

By the way, !

Thm 6.8: $\displaystyle\int_\Omega |f| \log(1+|f|)^n \, h(\log(1+|f|)) \, \omega^n < \infty$,

then $f \in \mathcal{F}(A,h)$ for some $A > 0$.

(So you do need $f \in L^{1+\varepsilon}(M)$ to solve MA, as Valentino said.)

(have existence!)

Thm (Stability): $f, g, 1 \in \mathcal{F}(A,h)$, $\omega_\varphi^n = f\omega^n$, $\omega_\psi^n = g\omega^n$.

Assume $\max_M (\varphi - \psi) = \max_M (\psi - \varphi)$.

There exists an increasing fun $\gamma(t)$ w/ $\lim_{t\to 0} \gamma(t) = 0$

for $t < t_0$,

$$\|f-g\|_1 \le \gamma(t) t^{n+3}$$

$$\Rightarrow \|\varphi - \psi\|_\infty \le (4a(3A,h)+2)\, t$$

$a(3A,h)$ ↳ unif. lower bound for $\mathcal{F}(A,h)$.

moreover,

$a := a(3A,h)$

$q = \left(\frac{3}{2}\right)^{1/n}$

$$\gamma(t) = \frac{(2q)^n}{(a+1)^n}\frac{q-1}{3} K_{A,h}^{(-1)}(t)$$

($K^{(-1)}$ → inverse fun.)

Proof: Denote $E_k := \{\psi < \varphi - kat\}$

$$C_0 := \int_{E_2} g\omega^n.$$

Step 1: bound C_0.

Assume wlog $\int_{E_0 = \{\psi < \varphi\}} (f+g)\omega^n \le 1$, o/w $f \leftrightarrow g$

Fix $t_0 < \frac{q-1}{2}$ and s.t. $\gamma(t_0) t_0^{n+3} < 1/3$.

$$\Rightarrow \int g\omega^n = \frac{1}{2}\int g+f+(g-f)\,\omega^n \le \frac{1}{2}\left(1 + \gamma(t_0)t_0^{n+3}\right) \le \frac{1}{2}\left(1+\tfrac{1}{3}\right) = 2/3.$$

(where $\int (g-f)\omega^n \le \|f-g\|_1$)

Define an ω-psh ρ solving $\omega_\rho^n = g_1 \omega^n$, $\max_M \rho = 0$

where $g_1 = \begin{cases} \frac{3}{2} g \text{ on } E_0 \\ c_0 > 0 \text{ otw} \end{cases}$ and $\int g_1 \omega^n = 1$

$c_0 \geq 0$ b/c had $\int_{E_0} g\omega^n \leq \frac{2}{3}$.

Now, $\forall E$ measurable,

$$\int_E g_1 \omega^n \leq \int_E \tfrac{3}{2} g + 1\ \omega^n \leq 3A / \mathrm{cap}_\omega(E)^{1+\alpha}$$

$\Rightarrow g_1 \in \mathcal{F}(3A, h)$ — defn of $g \in \mathcal{F}(A,h)$.

$\Rightarrow \rho \geq -a$.

Also arrange that $0 \geq \varphi \geq -a$. $\quad \varphi + c$, $\psi + c$

~~ρ is stly worse than φ.~~

Next: $\varphi - 2at \leq \varphi + t(\rho - \varphi - a) \leq \varphi$

$\therefore E_2 \subset \mathcal{E} := \{\psi < (1-t)\varphi + t\rho - at\} \subset E_0$

Let $G := \{f < (1-t^2) g\}$ (bad set.)

<u>Lemma</u>: $\omega_u^n, \omega_v^n \geq g\omega^n$

$\Rightarrow \omega_u^k \wedge \omega_v^{n-k} \geq g\omega^n$.

Pf $n=2$: $\ln\det(\lambda\omega_u + (1-\lambda)\omega_v) \geq \lambda \ln\det\omega_u + (1-\lambda)\ln\det\omega_v$

$\lambda^2 \ln\omega_u^2 + 2\lambda(1-\lambda) \ln\omega_u\wedge\omega_v + (1-\lambda)^2 \ln\omega_v^2 \geq$

$\lambda = \frac{1}{2}$ $\quad = \frac{1}{4}\ln\omega_u^2 + \frac{1}{4}\ln\omega_v^2 + \frac{2}{4}\ln\omega_u\wedge\omega_v \geq \ln\det\omega_u + \ln\det\omega_v$

On $E_0 \setminus G$: $\omega_\varphi^n = f\omega^n \geq (1-t^2) g\omega^n$, $\omega_\rho^{n-k} \geq q^{n-k} g\omega^n$

$$\Rightarrow \left(\frac{\omega_\varphi}{(1-t^2)^{1/n}}\right)^k \wedge \left(\frac{\omega_\rho}{q}\right)^{n-k} \geq g\omega^n$$

$\Rightarrow$ on $E_0 \setminus G$:

$$\omega^n_{t\rho+(1-t)\varphi} = \sum_{k=0}^n \binom{n}{k} (1-t)^k t^{n-k} \omega_\varphi^k \wedge \omega_\rho^{n-k}$$

$$\geq \left[(1-t)(1-t^2)^{1/n} + qt\right] g\omega^n \geq \left[(1-t)(1-t^2) + qt\right]^n g\omega^n$$

$$\geq \left[1 - t(q-1) - t^2\right] g\omega^n \geq \left[1 + \tfrac{t}{2}(q-1)\right] g\omega^n$$

$t(q-1-t)$

b/c $t < t_0 < \frac{q-1}{2}$

Note also that on G, from $f < (1-t^2) g$, we have

$$t^2 \int_G g\omega^n \leq \int_G (g-f) \leq \gamma(t) t^{n+3}$$

$$\Rightarrow \int_G g\omega^n \leq \gamma(t) t^{n+1}$$

So:

$$\left[1 + \tfrac{t}{2}(q-1)\right] \int_{E\setminus G} g\omega^n \leq \int_{E\setminus G} \omega^n_{(t\rho+(1-t)\varphi)} \leq \int_E \omega^n_{t\rho+(1-t)\varphi - at}$$

$$\overset{\text{C.P.}}{\leq} \int_E g\omega^n = \int_{E\setminus G} g\omega^n + \gamma(t) t^{n+1}$$

$$\Rightarrow \frac{(q-1)}{2} \int_{E\setminus G} g\omega^n \leq \gamma(t) t^n$$ almost s.t.p. ?

$$\Rightarrow \frac{q-1}{2}\left(C_0 - \gamma(t)t^{n+1}\right) \le \frac{q-1}{2}\left(\int_{E_2} g\omega^n - \int_G \omega^n\right)$$

$$= \frac{q-1}{2}\int_{E\setminus G} g\omega^n = \gamma(t)t^n.$$

$$\Rightarrow C_0 \le \Big(\overset{<1}{t} + \overset{>1}{\frac{2}{q-1}}\Big)\gamma(t)t^n \le \frac{3}{q-1}\gamma(t)t^n.$$

<u>Step 2</u>: turn this into a capacity bound on E_4, using <u>Lemma.</u> NEXT PAGE.

$$\mathrm{cap}_\omega(E_4) \le \frac{(a+1)^n}{(2at)^n}\int_{E_2} g\omega^n \quad (\le C_0)$$

$$= \frac{(a+1)^n}{(2a)^n}\frac{3}{q-1}\gamma(t).$$

<u>Step 4</u>: capacity bound $\Rightarrow$ slightly smaller set is empty (choice of γ.)

Suppose $E' = \{\psi < \varphi - 4a - 2t\} \neq \phi$. $\subset E_4$.

Then from (big) <u>lemma 6.5</u>,

$$2t \le K(\mathrm{cap}_\omega(E_4)) \le K\left(\frac{(a+1)^n}{(2a)^n}\frac{3}{q-1}\gamma(t)\right)$$

If we choose γ s.t. $K(-) = t$,
(K is invertible w/ $K(0)=0$)

this is a contradiction. Therefore

$$E' = \{\psi < \varphi - 4a - 2t\} = \phi$$

$$\Rightarrow \max(\varphi - \psi) < 4a + 2t \quad \square$$

$$= \max(\psi - \varphi) = \|\varphi - \psi\|_\infty.$$

If $h = x^n$, $K = (At)^{1/n}$
$\Rightarrow \gamma(t) = C\cdot t^n$, so choose t
s.t. $\|f-g\|_1 \doteq C t^n t^{n+3}$
$\Rightarrow \|\varphi - \psi\| \le C\|f-g\|_1^{1/(2n+3)}$

Appendix:

Lemma 3 = 6.10 $\quad 0 \le \varphi \le C-1, \quad s < C.$

$$\mathrm{cap}_\omega\{\psi + 2s < \varphi\} \le C^n s^{-n} \int_{\{\psi + s < \varphi\}} g\,\omega^n$$

Pf: Let $E(s) = \{\psi + s < \varphi\}$, $0 \le \rho \le 1$ PSH test fcn.

$$V = \{\psi < \tfrac{s}{C}(\rho - 1) + (1 - \tfrac{s}{C})\varphi - s\} \qquad -\frac{s(\rho - \varphi - 1 + \rho)}{C}$$

$$\psi + 2s < \varphi \Rightarrow \psi < \varphi - \tfrac{2s}{C} < \varphi - \tfrac{s}{C}\varphi + \tfrac{s}{C}(\rho - 1) - s$$

$$\Rightarrow \psi < \varphi - \frac{s(\varphi + 1 - \rho)}{C} - s < \varphi - s.$$

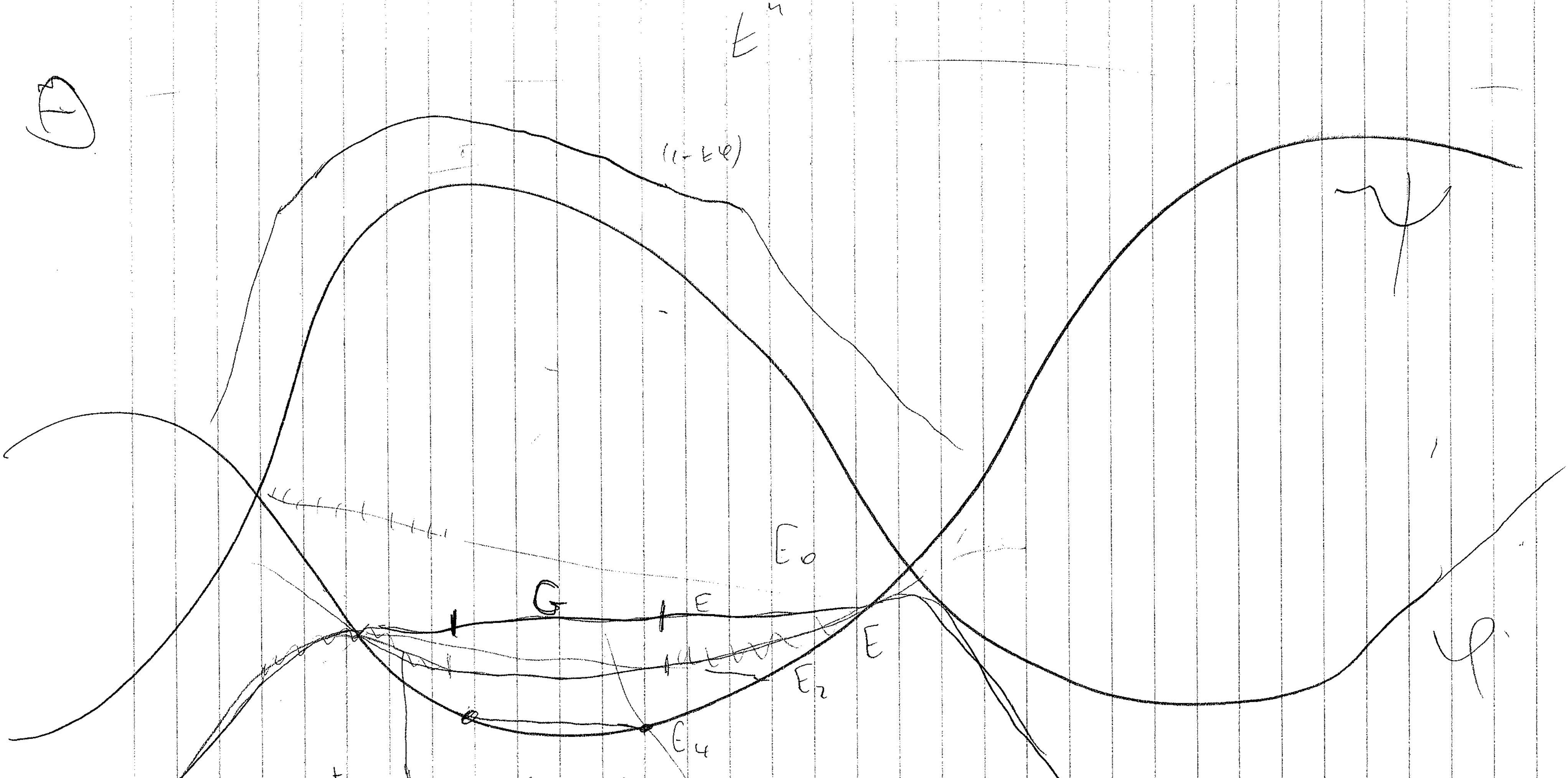
(almost $t = \frac{s}{C}$:)

$$\frac{s^n}{C^n}\int_{E(2s)} \omega_\rho^n \le \int_V \left(\frac{s}{C}\omega_\rho + (1 - \tfrac{s}{C})\omega_\varphi\right)^n$$

$$\le \int_V \omega_\varphi^n \le \int_{E(s)} g\,\omega^n. \qquad \square$$

Appendix 2

(1)



$E_0 \setminus G$: ψ^n greater than $C \cdot g\omega^n$ due to $f \geq (1-t^2)g$ = close enough to jack over from Newton (jacked-up) $3/2\, g$.

Otoh, from comparison on the whole E it must be lower than $g\omega^n$. So g must catch up within G, but L^1 controls that.

g-size of $G = \{f < (1-t^2)g\}$ controlled absolutely from L^1 bound. → So this gets f close enough that

So $\int_{E \setminus G} g\omega^n \leq \gamma(t) t^n \cdot C$.

Next: $\int_{E_2} g\omega^n \leq \frac{\xi}{t_n} \gamma(t)(t^n + t^{n+1})$

Need turn into a capacity!

Hence the E_4. Then to empty.