Semipositive (1,1)-Classes on Projective mfds

Theorem: (Nakai-Moishezon-Kleiman Criterion)

Let L be a line bundle on a projective scheme X. Then L is ample if and only if

$$\int_V c_1(L)^{\dim V} > 0$$

$\forall$ irreducible, positive dimensional subvarieties $V \subseteq X$.

In particular, a rational class $\alpha \in H^2(X, \mathbb{Q}) \cap H^{1,1}(X, \mathbb{R})$ is Kähler iff $\alpha^k \cdot Y > 0$ $\forall$ k-dimensional subvarieties $Y \subseteq X$.

The goal for this talk is to extend this to <u>Real</u> (1,1) classes.

Theorem 1: Let (X, ω) be a projective mfd and α a d-closed form of (1,1)-type, such that $\alpha \geq 1$ on X and $\alpha > 0$ at some point of X. Then $\exists$ a function $\varphi \in L^1(X)$ and some positive constant λ s.t. $\alpha + i\partial\bar{\partial}\varphi \geq \lambda\omega$ as currents on X.

Corollary: In the above situation, if $\alpha^k \cdot Y > 0$ $\forall$ $Y \subseteq X$ subvarieties, $\dim Y = k$, then $\{\alpha\}$ is a Kähler class.

Sketch of the pf.

Step 1: Fix an ample divisor H, and use Calabi-Yau theorem to construct "singular metric" with poles along H.
$$T = \alpha + i\partial\bar\partial\varphi$$

Step 2: Compute the Lelong number of T at a generic pt. of H. Show $\nu(T, x) \geq \lambda > 0$. ~~for~~

Step 3: Siu $\Rightarrow$ ~~we~~ $T \geq \lambda[H]$. But $\omega \in [H]$ is Kähler since H is ample, so $T \geq \lambda\omega$.

To go from Theorem to Corollary, need some other results of Paun. ~~from~~ ~~this~~

Step 1: Let $L \to X$ be a very ample line bundle, h a metric w/ curvature form $\omega > 0$, and let $H = (s = 0)$ for $0 \neq s \in H^0(X, L)$.

set $\psi_k(z) = \log\left(|s(z)|^2 + \frac{1}{k}\right)$.

Let $p_0 \in H$, and $P_n(p_0, 1)$ be a small disk, w/ radius 1 centered at p_0, w/ coordinates $(z_1, \ldots, z_n)$.

s.t. $H = (z_1 = 0)$, $\omega(p_0) = \sqrt{-1} \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i$

Trivialize L s.t. $h = e^{-\varphi(z)}$ w/ $\varphi(z) = |z|^2 + \ldots$

(possible since $\omega = i\partial\bar\partial\varphi$.) <u>Then</u>:

$$i\partial\bar\partial \psi_k = -\omega + i\partial\bar\partial \log\left(|z_1|^2 + \frac{e^{\varphi(z)}}{k}\right)$$

<u>and</u>

positive

$$(1)\quad i\partial\bar\partial \log\left(|z_1|^2 + \frac{e^{\varphi(z)}}{k}\right) = \frac{i\, dz_1 \wedge d\bar{z}_1 + \frac{1}{k} i\partial\bar\partial h^{-1}}{\left(|z_1|^2 + h/k\right)} - \frac{|z_1|^2 i\, dz_1 \wedge d\bar{z}_1}{\left(|z_1|^2 + \frac{1}{k}h^{-1}\right)^2}$$

$$\geq C^{-1} \frac{i\, dz_1 \wedge d\bar{z}_1}{k\left(|z_1|^2 + \frac{1}{k}\right)^2} \quad \text{after possibly shrinking the nbhd.}$$

<u>So</u> $\forall$ pts. z in a small tubular nbhd of H

$$(2)\quad \omega + i\partial\bar\partial\psi_k \geq C^{-1} \frac{dz_1 \wedge d\bar{z}_1}{k\left(|z_1|^2 + \frac{1}{k}\right)^2}$$

<u>And</u>: $\forall z \in X \setminus H$, $i\partial\bar\partial\psi_k \geq \dfrac{|S(z)|^2}{\left(|S(z)|^2 + \frac{1}{k}\right)} i\partial\bar\partial \log |S(z)|^2$

$$\geq -\lambda\omega \quad \text{for } \lambda < 1.$$

Thus $2\omega + i\partial\bar{\partial}\psi_k \geq \omega$. $\Rightarrow$ $2n + \Delta\psi_k \geq n$.

Consider

$$\left(\alpha + \frac{1}{k}\omega + i\partial\bar{\partial}\varphi_k\right)^n = (2n + \Delta\psi_k)\omega^n + \delta_k\left(\alpha + \frac{1}{k}\omega\right)^n \quad (3)$$

By Yau's Theorem, this has a solution φ_k when:

(i) The RHS is positive

(ii) $\int_X \left(\alpha + \frac{1}{k}\omega\right)^n = \int_X 2n(\omega)^n + \delta_k\left(\alpha + \frac{1}{k}\omega\right)^n$

of course, $\delta_k = \dfrac{\int_X \left(\alpha + \frac{1}{k}\omega\right)^n - 2n\int_X \omega^n}{\int_X \left(\alpha + \frac{1}{k}\omega\right)^n}$, so it remains only to ensure the RHS is positive. This follows immediately if $\delta_k \geq 0$. In order to ensure this, replace α w/ $C\alpha$ ~~st.~~ $\int_X \alpha^n \geq (2n)^2 \int_X \omega^n$. ~~we~~

we can do this since $\alpha > 0$ at some point, ~~st~~ and so $\int_X \alpha^n > 0$.

Now, we have unique φ_k solving (3), s.t. $\int_X \varphi_k \omega^n = 0$,
and $i\partial\bar\partial \varphi_k > -(\alpha+\omega) \Rightarrow |\varphi_k|_{L^1} \le C_1$ and $\sup_X \varphi_k \le C_1$ where $C_1 = C_1(X, \omega, \alpha)$ by the usual Greens formula trick.

Consider $T_k := \alpha + \frac{1}{k}\omega + i\partial\bar\partial\varphi_k$, $\begin{cases} T_k > 0 \\ dT_k = 0 \end{cases}$
and $\|\varphi_k\|_{L^1} < C_1$. So, weak compactness of currents implies $\exists\, m_k \to \infty$ and T s.t. $T_{m_k} \to T$. clearly T is a closed, positive current, in the class $\{\alpha\}$.
So, write $T = \alpha + i\partial\bar\partial\varphi$

Proposition: $\exists\, C > 0$ s.t. $\forall\, 0 < r << 1$ $\exists$ a set $W_\infty(r) \subseteq H$, with $Vol_H(W_\infty(r)) \geq C^{-1} Vol(H)$, and if $p \in W_\infty(r)$, we have

$$\frac{1}{r^{2n-2}} \int_{B(p,r)} T \wedge \omega^{n-1} \geq C^{-1}. \qquad (9).$$

Let's assume this and finish the pf.

<u>Claim</u>: for generic $x \in H$, the Lelong number $\nu(T,x) > 0$.

<u>pf</u> Assume <u>not</u>. Then for $x \in H \setminus \bigcup_i Z_i$ (Z_i analytic sets of measure zero) we have $\nu(T,x) = 0$. Let $K \subsetneq H$ be a cpct subset s.t. $\mathrm{Vol}(K) \geq (1 - \frac{c^{-1}}{2})\,\mathrm{Vol}(H)$ and $\nu(T,x) = 0$ $\forall x \in K$. Fix $z_0 \in K$.

<u>Then</u> $$\nu(T,x) = \lim_{r \to 0} \frac{1}{r^{2n-2}} \int_{B(z_0,r)} T \wedge \omega^{n-1} = 0.$$

Then $\exists\, r(z_0)$ s.t. $r \leq r(z_0)$ then $$r^{2-2n} \int_{B(z_0,r)} T \wedge \omega^{n-1} \leq \frac{c^{-1}}{2^{2n+1}}$$

<u>Now</u> if $z \in B(z_0, \frac{r(z_0)}{2})$, then $B(z, \frac{r(z_0)}{2}) \subseteq B(z_0, r(z_0))$ and so

$$\frac{1}{(r(z_0)/2)^{2n-2}} \int_{B(z, \frac{r(z_0)}{2})} T \wedge \omega^{n-1} \leq c^{-1}/2.$$

<u>Now</u>, the function $r \mapsto \frac{1}{r^{2n-2}} \int_{B(z,r)} T \wedge \omega^{n-1}$ is non-decreasing

and so $\forall z \in B(z_0, \frac{r(z_0)}{2})$ and $r \leq \frac{r(z_0)}{2}$

we have
$$\frac{1}{r^{2n-2}} \int_{B(z,r)} T \wedge \omega^{n-1} \leq \frac{C^{-1}}{2}.$$

Thus, by compactness $\exists\, r_0$ s.t. $\forall z \in K$, if $r \leq r_0$, then
$$\frac{1}{r^{2n-2}} \int_{B(z,r)} T \wedge \omega^{n-1} \leq C^{-1}/2. \qquad (10)$$

But, ~~for~~ $Vol(K) \geq (1 - \frac{C^{-1}}{2}) Vol(H)$ and for r small $Vol_*(W_\infty(r)) \geq C^{-1} Vol(H)$ so $K \cap W_\infty(r) \neq \emptyset$. But (10) and (9) are not compatible, so the claim follows.

Lemma (SM)

If T is a closed, positive $(1,1)$ current and H is a divisor, let $m_H := \inf_{z \in H} \nu(T,z)$. Then,
$$\nu(T,x) = m_H$$ for x in the compliment of a countable union of analytic sets in H.

In particular, $m_H > 0$ in our case. Next, we need

Prop 1: (Siu) Suppose T is a closed $(1,1)$ current, H is an irreducible analytic subset of X, $\dim_{\mathbb{C}} H = n-1$. If m_H is the generic Lelong number of T on H, then $T - m_H[H]$ is positive.

Thus $\alpha + i\partial\bar\partial\varphi \geq \lambda[H] = \lambda(\omega - i\partial\bar\partial\tau)$ and so the result follows. ∎

Pf of the proposition

$P_0 \in H$, $P_n(P_0, 1)$ polydisk of radius 1. Coordinates s.t. $H = (z_1 = 0)$. write

$$\alpha + \frac{1}{k}\omega + i\partial\bar\partial\varphi_k = i\sum_{s,q} \varphi_{k;s,q}\, dz_s \wedge d\bar z_q.$$

for $\varphi_{k;s,q}$ smooth. Set $\varphi_{k,j} := \varphi_{k;j,j}$

Note that $\alpha + \frac{1}{k}\omega + i\partial\bar\partial\varphi_k$ must Blow up along H as $k \to \infty$ by (2). we want to understand how. Note that

$$\int_X \left(\alpha + \frac{1}{k}\omega + i\partial\bar\partial\varphi_k\right) \wedge \left(\omega + i\partial\bar\partial\psi_k\right) \wedge \omega^{n-2} \leq C(\alpha, \omega)$$

and by (2) we have $\omega + i\partial\bar\partial\psi_k \geq \dfrac{c^{-1} i\, dz_1 \wedge d\bar z_1}{k(|z_1|^2 + \frac{1}{k})^2}$

so

$$\int_{P_n(P_0,1)} \left(\alpha + \frac{1}{k}\omega + i\partial\bar\partial\varphi_k\right) \wedge \frac{i\, dz_1 \wedge d\bar z_1}{k(|z_1|^2 + \frac{1}{k})^2} \wedge \omega^{n-2} \leq C(\alpha, \omega)$$

$$\Rightarrow \int_{P_n(P_0,1)} \sum_{j=2}^{n} \varphi_{k,j} \cdot \frac{1}{k(|z_1|^2 + \frac{1}{k})^2} \, d\lambda(z) \leq C.$$

Let $U_k := \left(|x_1|^2 < \frac{1}{k}\right) \times \left(|y_1|^2 < \frac{1}{k}\right) \times C_{n-1}(P_0, 1) \subseteq P_n(P_0, 1)$

where $z_1 = x_1 + i y_1$ and $C_{n-1}(P_0,1) = \prod_{j=2}^{n} (|x_j| < 1) \times (|y_j| < 1)$

Then

$$\int_{U_k} \sum_{j=2}^{n} \varphi_{k,j} \frac{1}{k(|z_1|^2 + \frac{1}{k})^2} \, d\lambda(z) \leq C.$$

Lemma: (X,μ) top. space, μ prob. measure, $f: X \to \mathbb{R}_+$ continuous, $\int_X f \leq C$. Then $\exists\, W \subseteq X$, $\int_W d\mu \geq \frac{1}{2}$ s.t. $f|_W \leq 2C$. (In fact $W = \{f \leq 2C\}^\circ$).

We apply this to $(X,\mu) := \left(U_k, \frac{d\lambda(z)}{k(|z_1|^2+\frac{1}{k})^2}\right) = (U_k, d\lambda_k)$ and $f := \sum_{j=2}^{n} \varphi_{k,j}$. Note that $\int_{U_k} d\mu = \frac{1}{k}\int_0^{1/k}\int_0^{1/k} \frac{dx\,dy}{(x^2+y^2+\frac{1}{k})^2}$

$=$

<u>Thus</u> we get $W_k \subseteq U_k$ open s.t.

(i) $\mathrm{Vol}_{\lambda_k}(W_k) \geq \frac{1}{2}\mathrm{Vol}_{\lambda_k}(U_k)$

(ii) $\varphi_{k,j}^{(w)} \leq C \quad \forall\ j = 2,\dots,n$ and $w \in W_k$.

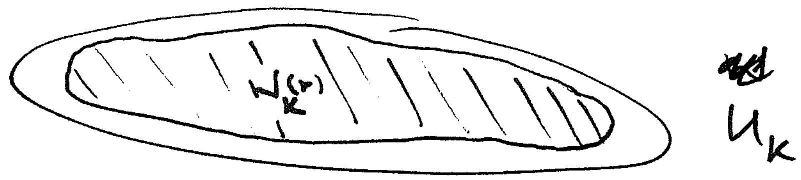
(not really, only diagonal entries)

In particular, all the "eigen values" other that the $\frac{\partial}{\partial z_1}$ component are bounded! So we have good control of the blow up in directions normal to H.

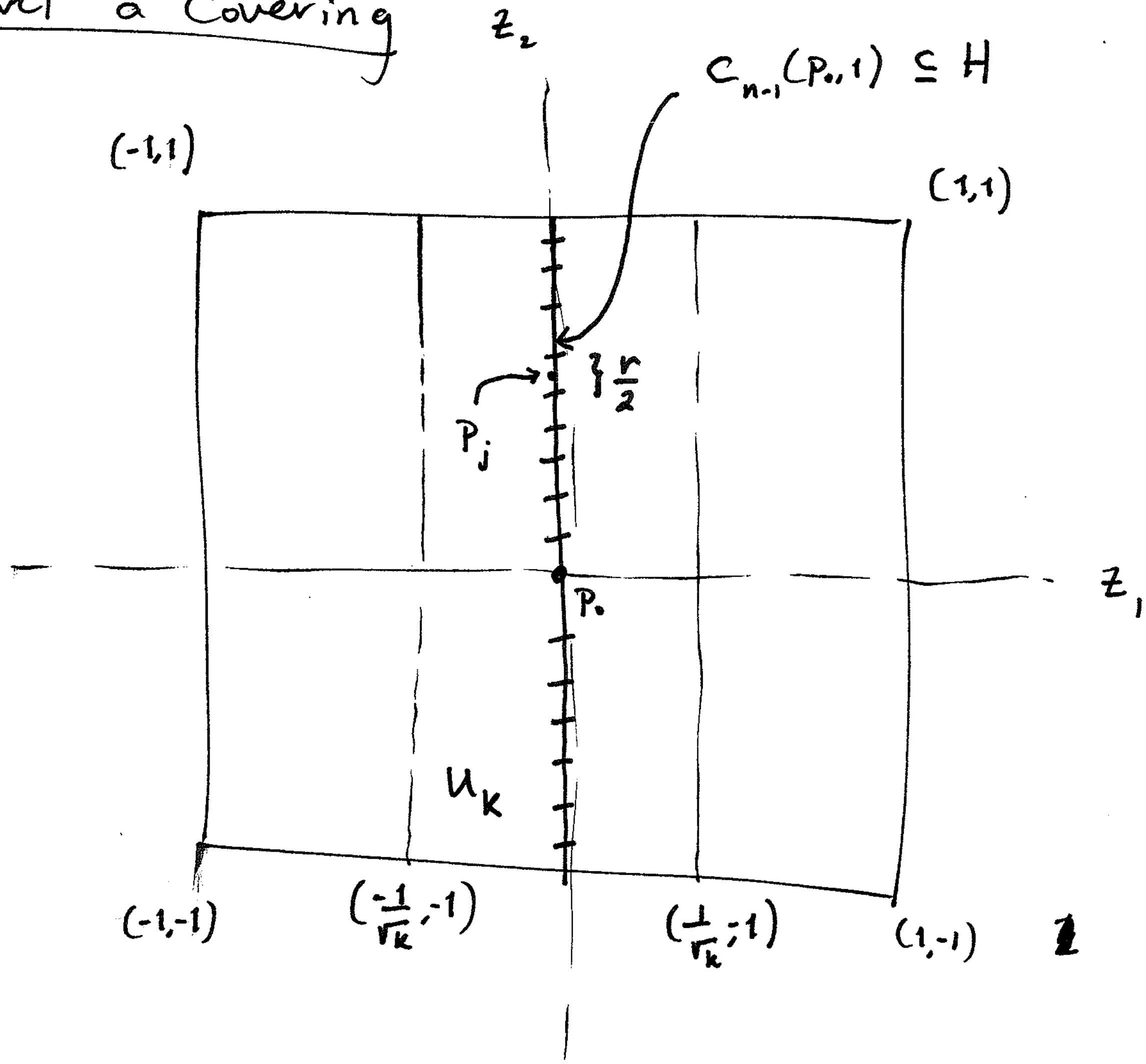
Fix $r << 1$, and let $k \geq k_0(r)$ (to be defined).
Set

$$W_k(r) := \left\{ p \in (z_1 = 0) : \mathrm{Vol}_{\lambda_k}\left(C_n(p,r) \cap W_k \right) \geq \frac{1}{2^{2n+2}} \mathrm{Vol}_{\lambda_k}\left(C_n(p,r) \cap U_k \right) \right\}$$

Here $C_n(p,r)$ = 2n dim'l Real cube of side length r.



Construct a Covering



cover $C_{n-1}(P_0, 1)$ with cubes of side length $r/2$ (assume $r = \frac{1}{m}$, $m \in \mathbb{Z}^+$) let P_j be the centres of the cubes. Then $\bigcup_j C_n(P_j, \frac{r}{2})$ covers U_k if $\frac{r}{k} > \frac{1}{\sqrt{k}}$

so take $k_0(r) \geq 2^4/r^2$. set

$$J := \left\{ j \in \mathbb{N} \;\middle|\; \mathrm{Vol}_{\lambda_k}\left(C_n(P_j, r/2) \cap W_k\right) \geq \frac{1}{4} \mathrm{Vol}_{\lambda_k}\left(C_n(P_j, r/2) \cap U_k\right) \right\}$$

~~For simplicity, suppress λ_k~~

By construction of the covering

$$\mathrm{Vol}_{\lambda_k}(U_k) \leq \mathrm{Vol}_{\lambda_k}\left(\bigcup_j C_n(P_j, \tfrac{r}{2}) \cap U_n\right) = \cancel{\sum_j \left[\frac{1}{k} \int_0^{\frac{1}{k}} \int_0^{\frac{1}{k}} \frac{dx\,dy}{(x^2+y^2+\frac{1}{k})^2}\right] C_{n}(P_j, \tfrac{r}{2})}$$

$$= \sum_j \left[\frac{1}{k} \int_0^{\frac{1}{k}} \int_0^{\frac{1}{k}} \frac{dx\,dy}{(x^2+y^2+\frac{1}{k})^2}\right] \mathrm{Vol}_{\lambda}\left(C_{n-1}(P_j, r/2)\right)$$

$$= \tilde{C} \cdot (r/2)^{2n-2} \left(|J| + |J^c|\right)$$

<u>But</u> By def'n of J:

$$\mathrm{Vol}_{\lambda_k}\left(\bigcup_{j \in J^c} C_n(P_j, r/2) \cap W_k\right) \leq \frac{1}{4} \tilde{C} \left(\frac{r}{2}\right)^{2n-2} |J^c|$$

and

$$\mathrm{Vol}\left(\bigcup_{j\in J} C_n(p_j,\tfrac{r}{2})\cap W_k\right) \le \left(\tfrac{r}{2}\right)^{2n-2} |J|\,\tilde{C}.$$

Thus $\tilde{C}\left(|J| + \frac{1}{4}|J^c|\right)\left(\frac{r}{2}\right)^{2n-2} \ge \mathrm{Vol}(W_k) \ge \frac{1}{2}\mathrm{Vol}(U_k) = \frac{1}{2}\tilde{C}\left(\frac{r}{2}\right)^{2n-2}\left(|J|+|J^c|\right)$

$$\Rightarrow \quad |J| + \tfrac{1}{4}|J^c| \ge \tfrac{1}{2}\left(|J|+|J^c|\right)$$

so $\quad |J| \ge \frac{1}{3}\left(|J|+|J^c|\right)$. so

$$\mathrm{Vol}_H\left(\bigcup_{j\in J} C_n(p_j,\tfrac{r}{2})\cap H\right) = \left(\tfrac{r}{2}\right)^{2n-2} |J| \ge \tfrac{1}{3}\mathrm{Vol}_H\left(C_{n-1}(p_0,1)\right)$$

where $\mathrm{Vol}_H(A)$ = Lebesgue measure of $A \subseteq H$.

But if $p \in C_n(p_j, r/2)\cap H$, then

$$\mathrm{Vol}_{\lambda_k}\left(C_n(p,r)\cap W_k\right) \ge \mathrm{Vol}_{\lambda_k}\left(C_n(p_j,r/2)\cap W_k\right) \ge \tfrac{1}{4}\mathrm{Vol}_{\lambda_k}\left(C_n(p_j,\tfrac{r}{2})\cap U_k\right)$$

$$= \frac{1}{2^{2n+2}}\mathrm{Vol}_{\lambda_k}\left(C_n(p,r)\cap U_k\right) \quad \text{so}$$

$$W_k(r) \supseteq \bigcup_{j\in J} C_n(p_j,\tfrac{r}{2})\cap H \quad \text{and Hence}$$

$$\mathrm{Vol}\left(W_k(r)\right) \ge c \quad \text{independent of } k, r.$$

The defining equation (3) for φ_k gives

$$\det(\varphi_{k;s,\bar q}) \geq C^{-1} \frac{1}{k(|z_1|^2+\frac{1}{k})^2}.$$

Then linear algebra yields $\prod_{j=1}^{n} \varphi_{k,j}(z) \geq C^{-1} \frac{1}{k(|z_1|^2+\frac{1}{k})^2}$

<u>and so</u>: for $p \in W_k(r)$

$$\int_{C_n(p,r)\cap W_k} \prod_{j=1}^{n} \varphi_{k,j}(z)\, d\lambda(z) \geq \mathrm{Vol}_{\lambda_k}(C_n(p,r)\cap W_k) \geq C^{-1} r^{2n-2}$$

<u>But</u> by property (ii) of W_k we get

$$\int_{C_n(p,r)\cap W_k} \prod_{j=1}^{n} \varphi_{k,j}(z)\, d\lambda \leq C \int_{C_n(p,r)\cap W_k} \varphi_{k,1}(z)\, d\lambda$$

$$\leq C \int_{C_n(p,r)} \sum_{j=1}^{n} \varphi_{k,j}(z) = C\int_{C_n(p,r)} \left(\alpha + \tfrac{1}{k}\omega + i\partial\bar\partial\varphi_k\right) \wedge \left(i\partial\bar\partial|z|^2\right)^{n-1}$$

<u>Thus</u> if $p \in W_k(r)$ $\displaystyle\int_{C_n(p,r)} \left(\alpha + \tfrac{1}{k}\omega + i\partial\bar\partial\varphi_k\right) \wedge \left(i\partial\bar\partial|z|^2\right)^{n-1} \geq C^{-1} r^{2n-2}$

Now, take $\lim_{k \to \infty}$ and set $W_\infty(r) = \limsup_{k \geq k_0(r)} W_k(r)$.

<u>Thm</u> $W_\infty(r) \geq C^{-1} \operatorname{Vol}(H)$ and $\forall\, p \in W_\infty(r)$ we have

$$\int_{C_n(p,r)} T \wedge \omega^{n-1} \geq C^{-1} r^{2n-2}.$$

$\blacksquare$