Talk

* Plan • prove α- invariant exists

• ~~motivation from KE, $c_1(X)>0$~~

• ~~other related estimates.~~

X cpt Kähle

Recall $Psh(X,\omega) := \{ \varphi \in L^1(X),\ \varphi : X \to [-\infty, \infty),\ \omega + i\partial\bar\partial\varphi \geq 0 \}$ upper semi-cont.

by normalizing

Let $\alpha(X,[\omega]) = \sup\{ \alpha > 0 \mid \exists\, C>0 \text{ s.t. } \int_X e^{-\alpha\varphi}\, dV \leq C \ \ \forall \phi \in Psh \}$

ind. of Kähler class

the key is the bound holds for all ϕ

let's prove that such an α always exists

First, local result.

Work on a ball $B_1 \subset \mathbb{C}^1$. Recall from Dan's talk

; $\partial\bar\partial\phi \geq 0$ is a positive distribution

$\Rightarrow$ $\Delta\phi$ is a positive measure

Prop Assume ϕ is subharmonic in $B_2(0) \subset \mathbb{C}$ and $\phi(0) \geq 0$ and $\phi(z) < 1$ for $|z|<1$

Then $\int_{|z|<\frac{1}{2}} e^{-\phi(z)}\, dz \leq C$ ← universal independent of ϕ

<u>Proof</u> by Riesz Representation, for any $z \in B_1(0)$

$$2\pi\, \phi(z) = \int_{|\zeta|<1} \log \frac{|z-\zeta|}{|1-z\bar{\zeta}|}\, \Delta\phi \, d\zeta + \int_{|\zeta|=1} \frac{1-|z|^2}{|z-\zeta|^2}\, \phi \, \underset{\substack{\uparrow \\ \text{measure on } |\zeta|=1}}{d\sigma(\zeta)}$$

plug in $z=0$

$$0 \le \int_{|\zeta|<1} \log|\zeta|\, \Delta\phi \, d\zeta + \int_{|\zeta|=1} \phi \, d\sigma$$

$$\Rightarrow \underbrace{\int_{|\zeta|<1} \log\frac{1}{|\zeta|}\, \Delta\phi \, d\zeta}_{A \,\ge\, 0} + \underbrace{\int_{|\zeta|=1} (1-\phi)\, d\sigma}_{B \,\ge\, 0} \le 2\pi$$

$$\Rightarrow A \le 2\pi, \quad B \le 2\pi \Rightarrow \int_{|\zeta|=1} \phi \, d\sigma \ge 0$$

Now since $\phi \le 1$, we have $\int_{|\zeta|=1} \phi\, d\sigma \le 2\pi$

Upper bound plus $0 \le \int_{|\zeta|=1} \phi \, d\sigma \le 2\pi$

$$\Rightarrow \int_{\substack{\phi<0 \\ |\zeta|=1}} (-\phi)\, d\sigma \le 2\pi \quad \text{and} \quad \int_{\substack{\phi \ge 0 \\ |\zeta|=1}} \phi \, d\sigma \le 2\pi$$

$$\Rightarrow \int_{|\zeta|=1} |\phi| \, d\sigma \leq 4\pi$$

Now $\quad |z|<\frac{1}{2}$ and $|\zeta|=1 \Rightarrow \frac{1}{2} < 1-|z|^2 \leq 1$

and $\quad |z-\zeta| \geq |\zeta| - |z| = 1-|z| \geq \frac{1}{2}$

triang $\neq$

thus $\quad \frac{1}{|z-\zeta|^2} \leq 4$

So $\quad 0 \leq \frac{1-|z|^2}{|z-\zeta|^2} \leq 4$, thus

$$\star \quad \left| \frac{1}{2\pi} \int_{|\zeta|=1} \frac{1-|z|^2}{|z-\zeta|^2} \phi \, d\sigma(\zeta) \right| \leq \frac{4}{2\pi} \int_{|\zeta|=1} |\phi| \, d\sigma \leq 8$$

Note if ϕ is harmonic we're done. Assume not and keep going

pick $\quad |z|<\frac{1}{2}<R<|\zeta|<1$

$$0 < a := \int_{|\zeta|<R} \frac{\Delta\phi}{2\pi} d\zeta \leq \int_{|\zeta|<R} \frac{\log 1/|\zeta|}{\log \frac{1}{R}} \frac{\Delta\phi}{2\pi} \underset{\substack{\uparrow \\ \text{since } (A) \leq 2\pi}}{\leq} \frac{1}{\log 1/R} \frac{2\pi}{2\pi} < 2 \quad \text{if } R<e^{-\frac{1}{2}}$$

this is possible since $e<4$ $\quad \left(\frac{1}{2}<R<e^{-\frac{1}{2}}\right)$ $\quad$ Fix R !

If $\quad |z|<\frac{1}{2}<R<|\zeta|<1$

Claim then $\quad \log \frac{|1-z\bar{\zeta}|}{|z-\zeta|} \leq \underset{\text{universal}}{C} \log \frac{1}{|\zeta|}$

Given claim $\log \dfrac{|1-z\bar{\zeta}|}{|z-\zeta|} \leq C \log \dfrac{1}{|\zeta|}$

then $$\int_{R<|\zeta|<1} \log \frac{|1-z\bar{\zeta}|}{|z-\zeta|} \Delta\phi \, d\zeta \leq C \int_{|\zeta|<1} \log \frac{1}{|\zeta|} \Delta\phi \, d\zeta \overset{\text{by (A)}}{\leq} C \, 2\pi$$

by and ✱

Then $$e^{-\phi(z)} \leq e^{\left(\frac{1}{2\pi} \int_{|\zeta|<R} \log \frac{|1-z\bar{\zeta}|}{|z-\zeta|} \Delta\phi \, d\zeta + C\right)}$$

$$\leq C e^{\left(\int_{|\zeta|<R} a \log \frac{|1-z\bar{\zeta}|}{|z-\zeta|} \frac{\Delta\phi}{2\pi a} d\zeta\right)}$$

Jensen
$$\leq C \int_{|\zeta|<R} \left|\frac{1-z\bar{\zeta}}{z-\zeta}\right|^a \frac{\Delta\phi}{2\pi a} d\zeta \leq C \int_{|\zeta|<R} |z-\zeta|^{-a} \frac{\Delta\phi}{2\pi a} d\zeta$$

Now integrate

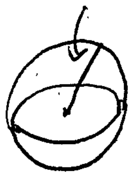
important that $\boxed{a<2}$

$$\int_{|z|<\frac{1}{2}} e^{-\phi(z)} dz \leq C \int_{|\zeta|<R} \left(\int_{|z|<\frac{1}{2}} |z-\zeta|^{-a} dz \right) \frac{\Delta\phi}{2\pi a} d\zeta$$

$$\leq C$$

complex lines

Completes proof when $n=1$. for $n>1$ switch to polar.



$$\int_{|z|<\frac{1}{2}} e^{-\phi(z)}\,dz = \int_{|\xi|=1} dS(\xi) \int_{|\omega|<\frac{1}{2}} |\omega|^{2n-2} e^{-\phi(\omega\xi)} \frac{d\omega}{2\pi}$$

$\boxed{\text{completes proof}}$

$\boxed{\text{Scaling}}$

Now in $B_R(0)$, assume $\phi(0) \geq -1$ and $\phi(z)<0$

then $\exists\ \lambda, C$ s.t $\displaystyle\int_{|z|<r} e^{-\lambda\phi(z)}\,dz \leq C$ where $r < Re^{-\frac{\lambda}{2}}$

$\boxed{\text{Global Version}}$

Prop

$\exists\ \alpha, C$ s.t $\displaystyle\int_X e^{-\alpha\phi}\,dV \leq C \quad \forall\ \phi \in Psh$

proof

Since $\phi \in Psh$, $\omega + i\partial\bar\partial\phi \geq 0 \Rightarrow -\Delta\phi \leq n$

Also $\displaystyle\phi(x) = \frac{1}{V}\int_X \phi\,dV - \int_X G(x,y)\,\Delta\phi\,dV(y)$

$\uparrow$ green's function

so

$$0 = \sup_X \phi \leq \frac{1}{V}\int_X \phi\,dV + \sup_X \int_X G(x,y)\,(-\Delta\phi)\,dV(y)$$

$$\leq \frac{1}{V}\int_X \phi\,dV + n \sup_X \int_X G(x,y)\,dV(y)$$

Let $X = \bigcup_{i=1}^{N} B_{r/4}(x_i)$ ← geodesic ball radius $\frac{r}{4}$

Using $\frac{1}{V}\int_X \phi(y)\, dV \geq - \sup \int_X G(x,y)\, dV(y) = -C$

Then $\sup_{B_{r/4}(x_i)} \phi(y) \geq \frac{1}{Vol(B)} \int_B \phi(y)\, V(y)$.

$$\geq - \frac{VC}{Vol(B)}$$

) since $\phi \leq 0$ no one ball can get too neg.

Let ψ_i be kähler potential on $B_{2r}(x_i)$

s.t. $\psi_i(x_i) = 0$, let $C_1 = \sup_i \sup_{x \in B_{\frac{3r}{2}}} |\psi_i(x)|$

Then $\psi_i(x) + \phi(x) \leq C_1$ in $B_{\frac{3}{4}r}(x_i)$

Then there exists a y_i in $B_{r/4}(x_i)$ s.t.

$$\phi(y_i) \geq \frac{-VC}{Vol(B_{r/4})}$$

set $\alpha = C_1 + \min_i \frac{Vol(B_{r/4}(x_i))}{VC_* + 1}$

$$\Rightarrow \int_{B_{r/2}(y_i)} e^{-\alpha(\psi_i(x)+\phi(x) - C_2)} dV \leq C$$

$$\Rightarrow \int_X e^{-\alpha\phi} dV \leq C$$

So ~~...~~ is non empty!

$\{\alpha > 0 \mid \exists C > 0 \text{ s.t. } \int_X e^{-\alpha\phi} \leq C \quad \forall \phi \text{ psh}\}$

α well defined

Thm (Tian) Let X be a Kähler manifold

st. ~~...~~ $\frac{1}{\pi}\omega$ represents $c_1(X)$

If $\alpha > \frac{n}{n+1}$, X admits K.E.

Cor 1 ~~...~~ n dim Fermat hypersurfaces with deg $> n-1$