MATH 263: PROBLEM SET 1: BUNDLES, SHEAVES AND HODGE THEORY

0.1. **Vector Bundles and Connection 1-forms.** Let $E \to X$ be a complex vector bundle of rank r over a smooth manifold. Recall the following "abstract" definition

Definition 0.1. *A connection on E is a $\mathbb{C}$-linear map on the space of smooth sections $\nabla : \Gamma(E) \to \Gamma(E \otimes \Lambda^1)$ satisfying the Leibniz rule*

$$\nabla(fs) = df \otimes s + f\nabla(s)$$

for any smooth section $s \in \Gamma(E)$, and any smooth function f.

(1) Given an open set $U \subset X$, and a local trivialization (that is, a local frame $\{e_\alpha\}_{1\leqslant\alpha\leqslant r}$) we get an isomorphism

$$E|_U \cong U \times \mathbb{C}^r$$

by writing a section $s \in \Gamma(U, E)$ as $s = s^\alpha e_\alpha$ and mapping $s \mapsto (s^\alpha) \in \mathbb{C}^r$. Show that we can write a connection on E as $\nabla = d + A$ where we can view A (locally) as a 1-form valued in $End(E) := Hom(E, E)$. That is we can write

$$A = A_{j\ \beta}^{\alpha} dx^j$$

where, for each j, $A_{j\ \beta}^{\alpha} : \mathbb{C}^r \to \mathbb{C}^r$. It is common to write d_A for the covariant derivative $d + A$.

(2) Work out the transformation rule for A under a change of frame. That is, if we take a new frame $\{\sigma_\alpha\}$ related by

$$\sigma_\alpha = t_\alpha^\beta e_\beta$$

for a map $t : U \to Gl(n, \mathbb{C})$, find an expression for the connection ∇ written in the frame σ_α. The key point here is that $\nabla(s)$ is independent of the choice of trivialization.

(3) Show that the curvature $F_\nabla := [\nabla, \nabla]$ of (E, ∇) can be regarded as a section $F_\nabla \in \Gamma(\mathrm{End}(E) \otimes \Lambda^2)$ which can be expressed as

$$F_\nabla s = d_A^2(s) = (dA + A \wedge A)s.$$

Prove the **second Bianchi identity**: $d_A F = 0$ (recall how that covariant derivative extends to $End(E)$).

(4) If X is a complex manifold, and E is equipped with a Hermitian metric H, let ∇ be the unitary Chern connection. Show that F_∇ is a section of $End(E) \otimes \Lambda^{1,1}$. That is, the $(2,0)$ and $(0,2)$ parts of the curvature vanish.

0.2. **Sheaf Theory.** Let X be a complex manifold, and let $\mathcal{F}$ be a sheaf of Abelian groups over X.

(1) Let $\underline{U} := \{U_\alpha\}_{\alpha \in A}$ be a locally finite open cover of X, and let $C^p(\underline{U}, \mathcal{F})$ denote the p-cochains. Recall we defined a coboundary operator

$$\delta : C^p(\underline{U}, \mathcal{F}) \to C^{p+1}(\underline{U}, \mathcal{F}).$$

Show that: $C^p(\underline{U}, \mathcal{F})$ has the structure of an abelian group, and that δ is a group homomorphism with $\delta^2 = 0$.

(2) Recall the following definition

Definition 0.2. *A (locally finite) open cover* $\underline{W} = \{W_\beta\}_{\beta \in B}$ *is called a refinement of* $\underline{U} = \{U_\alpha\}_{\alpha \in A}$ *is there is a map* $\mu : B \to A$ *such that*

$$W_\beta \subset U_{\mu(\beta)}.$$

The map μ *is referred to as the refining map.*

Show that μ defines a group homomorphism $\hat{\mu} : C^p(\underline{U}, \mathcal{F}) \to C^p(\underline{W}, \mathcal{F})$, and that $\delta\hat{\mu} = \hat{\mu}\delta$. Deduce that we get a map

$$\mu^* : H^p(\underline{U}, \mathcal{F}) \to H^p(\underline{W}, \mathcal{F})$$

(3) Suppose we have two refining maps $\mu, \nu : B \to A$. Define a map

$$\Theta : C^p(\underline{U}, \mathcal{F}) \to C^{p-1}(\underline{W}, \mathcal{F})$$

by

$$\Theta f(W_0, \dots, W_{p-1}) = \sum_{j=0}^{p-1} (-1)^j f(U_{\mu(0)}, \dots, U_{\mu(j)}, U_{\nu(j)}, \dots, U_{\nu(p-1)})|_{W_0 \cap \dots \cap W_{p-1}}.$$

Show that this map makes sense, and compute that

$$\hat{\mu}(f) = \hat{\nu}(f) + \delta\theta f.$$

In particular, conclude that $\mu^* = \nu^*$ as maps on cohomology. Deduce that Čech cohomology is well-defined.

(4) Show that if $\mathcal{F}$ is fine, then $\check{H}^p(X, \mathcal{F}) = 0$ for all $p \geqslant 1$.

0.3. **Hodge Theory Part 1: Linear Elliptic Operators.** To begin, let's recall some of the basic theory of Sobolev spaces, weak derivatives and elliptic operators. A good reference for this material is L.C. Evans- *Partial Differential Equations*, chapter 5 and chapter 6 . If you are unfamiliar with this subject, I recommend reading these sections.

Let $\Omega \subset \mathbb{R}^n$ be a domain. For functions $u, v \in L^1_{loc}(\Omega)$, and a multi-index α we say that v is the α-th weak derivative of u, written $D^\alpha u = v$, if

$$\int_\omega (-1)^{|\alpha|} u D^\alpha \varphi dx = \int_\Omega v\varphi dx$$

for any smooth function φ with compact support in Ω (ie. $\varphi \in C_0^\infty(\Omega)$). For the most part we will be interested in functions with just one weak derivative, which we denote by $Du = (D_1 u, , \dots, D_n u)$.

Definition 0.3. *The Sobolev space $W^{k,p}(\Omega)$ is defined to be the space of functions $u \in L^p(\Omega)$ such that $D^\alpha u$ exists, and $D^\alpha u \in L^p$ for all $|\alpha| \leqslant k$. This space has a norm given by*

$$\|u\|_{W^{k,p}(\Omega)} := \|u\|^{L^p(\Omega)} + \sum_{i=1}^{k} \|D^k u\|_{L^p(\Omega)}.$$

Theorem 0.4. *Some basic properties of the Sobolev spaces are:*

(i) *$W^{k,2}(\Omega)$ is a Hilbert space. It is common to denote $W^{k,2} = H^k$.*
(ii) *$W^{k,p}(\Omega)$ is a Banach space (Evans, section 5.2, Theorem 2).*
(iii) *If Ω is bounded, then $C^\infty(\Omega)$ is dense in $W^{k,p}(\Omega)$. We can then define $W_0^{k,p}(\Omega)$ to be the closure of the compactly supported smooth functions in $W^{k,p}(\Omega)$ (Evans, section 5.3, Theorem 2).*

Consider a differential operator on Ω

$$Lu := -D_i(a^{ij}(x) D_j u) + b^i(x) D_i u + cu$$

where we are summing over repeated indices. We say that $u \in W^{1,2}(\Omega)$ solves $Lu = f$ weakly if

$$\int_\Omega a^{ij} D_j u D_i \varphi + b^i (D_i u)\varphi + cu\varphi dx = \int_\Omega f\varphi dx \tag{0.1}$$

for all $\varphi \in C_0^\infty(\Omega)$.

(1) Show that if $u \in C^2(\Omega)$ solves $Lu = f$ pointwise, then $Lu = f$ weakly also.
(2) Show that (0.1) also holds for $\varphi \in W_0^{1,2}(\Omega)$.

We say that L is uniformly elliptic if $a^{ij} = a^{ji}$ and there exists a constant $\lambda > 0$ so that

$$\sum_{i,j} a^{ij}(x)\xi_i\xi_j > \lambda|\xi|^2$$

a.e. in Ω, for all $\xi \in \mathbb{R}^n$.

(3) Prove the following (fundamental) theorem, referred to commonly as "Interior Elliptic Regularity".

Theorem 0.5. *Let Ω be a bounded domain. Assume $a^{ij} \in C^1(\Omega)$, $b^i, c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$, and $u \in W^{1,2}(\Omega)$ solving*

$$Lu = f$$

in Ω, with L uniformly elliptic. Then $u \in W^{2,2}_{loc}(\Omega)$, and, for each $V \Subset \Omega$ we have

$$\|u\|_{W^{2,2}(V)} \leqslant C\left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right)$$

for a constant C depending only on Ω, V and the coefficients of L.

Here is a sketch for how to do this.

Step 1 Assume u is smooth, and $b^i = c = 0$ to begin with. Choose a direction ℓ, and apply the definition of a weak solution (that is, integration by parts) to the test function $\varphi := -D_\ell(\eta^2 D_\ell u)$ where η is a smooth function $\eta : \Omega \to [0, 1]$ with $\eta \equiv 1$ in V, and $\eta \equiv 0$ in a neighborhood of $\partial\Omega$. We can choose η so that $|D\eta| \leqslant 10\text{dist}(V, \partial\Omega)$. Use integration by parts and the inequality

$$|a||b| \leqslant \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2$$

to prove that

$$\lambda \int_\Omega |Du_\ell|^2\eta^2 \leqslant \left|\int_\Omega f\varphi\right| + C\int_\Omega |Du|^2 + \varepsilon\int_\Omega |Du_\ell|^2\eta^2$$

where C depends only on the C^1 norm of a^{ij}, ε and $\text{dist}(V, \partial\Omega)$. Here I have used $u_\ell := D_\ell u$.

Step 2 Estimate

$$\int_\Omega |f\varphi| \leqslant \varepsilon\int_\Omega \eta^2|Du_\ell|^2\eta^2 + C\int_\Omega |f|^2 + C\int_\Omega |Du|^2$$

with C, depending only on the data in step 1.

Step 3 By choosing ε appropriately, obtain the estimate

$$\int_V |Du_\ell|^2 \leqslant C\left(\|f\|^2_{L^2(\Omega)} + \|u\|^2_{W^{1,2}(\Omega)}\right)$$

with C depending on the C^1 norm of a^{ij}, $\text{dist}(V, \partial\Omega)$ and a lower bound for λ.

Step 5 Prove the same thing for general L by writing $Lu = f$ as

$$-D_i(a^{ij}(x)D_j u) = \tilde{f} = f - b^i(x)D_i u - cu.$$

(One only needs a bound for the L^2 norm of $\tilde{f}$ in terms of $\|f\|_{L^2}$ and $\|u\|_{W^{1,2}}$.)

Step 6 Now deduce the same thing for non-smooth functions. If $Lu = f$, let u_m be a sequence of smooth functions converging to u in $W^{1,2}(\Omega)$. Show that

$$Lu_m = f_m$$

where $f_m \in L^2(\Omega)$, and f_m converges to f in $L^2(\Omega)$. In particular, we have $\|f_m\|_{L^2(\Omega)} \to \|f\|_{L^2(\Omega)}$.

Step 7 Now by the estimate in the smooth case we get

$$\|u_m\|_{W^{2,2}(V)} \leqslant C\left(\|f_m\|_{L^2(\Omega)} + \|u_m\|^2_{W^{1,2}(\Omega)}\right),$$

for a uniform constant C. Since $W^{2,2}(V)$ is a Hilbert space, we conclude that u_m converges weakly (along some subsequence) to a limit u_∞ in $W^{2,2}(V)$. Since u_m converges to u strongly in $W^{1,2}(V)$ we know that $u_\infty = u$ in $W^{1,2}(\Omega)$. Show that D^2u_∞ is a weak second derivative of u, and hence $u \in W^{2,2}(V)$. Show that in a Hilbert space, the norm is lower semi-continuous along weak limits. That is, if x_n converges weakly to x in a Hilbert space H then

$$\|x\| \leqslant \liminf \|x_n\|.$$

Conclude that

$$\|u\|_{W^{2,2}(V)} \leqslant \left(\|f\|_{L^2(\Omega)} + \|u\|^2_{W^{1,2}(\Omega)}\right)$$

(4) The Sobolev embedding theorem implies, for example, that if Ω is a ball, and $u \in W^{k,p}(\Omega)$ for $kp > n$, then in fact $u \in C^{k-1-[n/p]}(\Omega)$ where $[n/p]$ denotes the integer part on n/p. As an application of the elliptic regularity theorem, show that if a^{ij}, b^i, c and f are smooth, and $Lu = f$ for some $u \in W^{1,2}(\Omega)$, then in fact $u \in C^\infty(\Omega)$. (**Hint**: Differentiate the equation and apply elliptic regularity repeatedly. This is referred to as "bootstrapping").

0.4. **Hodge Theory Part 2: Hodge Theory.** We are now in a position to prove Hodge's theorem. Let $E \to X$ be a holomorphic vector bundle with a Hermitian metric H on E, and a Hermitian metric g on X (not necessarily Kahler). Let $\overline{\Box} = \overline{\partial}\,\overline{\partial}^\dagger + \overline{\partial}^\dagger\,\overline{\partial}$ on $C^\infty(X, E \otimes \Lambda^{p,q})$.

Theorem 0.6. *Let $L^2(X, E\otimes\Lambda^{p,q})$ denote the space of L^2 (p,q) forms with values in E (I will suppress the bundle from now on). Then*

(a) *There is an orthonormal basis $\{\psi_\ell\}$ of L^2, $\psi_\ell \in C^\infty$ and with*

$$\overline{\Box}\psi_\ell = \lambda_\ell \psi_\ell$$

for $\lambda_\ell \in \mathbb{R}$.

(b) *There is an operator $G : L^2 \to L^2$, called the Green's operator, which is bounded and self-adjoint so that*

$$\overline{\Box} G = Id - \pi$$

where $\pi : L^2 \to L^2$ is the orthogonal projection onto the closed subspace

$$\mathrm{Ker}\overline{\Box} := \{\varphi \in \mathrm{C^\infty(X, E \otimes \Lambda^{p,q}} | \overline{\Box}\varphi = 0\}.$$

Furthermore, there is an L^2 orthogonal decomposition

$$L^2 = \mathrm{Ker}\overline{\Box} \oplus \mathrm{Range}\overline{\partial} \oplus \mathrm{Range}\overline{\partial}^\dagger.$$

The corollary of this that we are interested in is:

Corollary 0.7. *If $\varphi \in C^\infty(X, E \otimes \lambda^{p,q})$ as $\overline{\partial}\varphi = 0$, so that $[\varphi]$ defines a Dolbeaut (and hence Čech) cohomology class, then $[\varphi]$ can be represented by a harmonic representative $\pi\varphi \in \mathrm{Ker}\overline{\Box}|_{E\otimes\lambda^{p,q}}$.*

Note that $\mathrm{Ker}\overline{\Box}|_{E\otimes\Lambda^{p,q}}$ seems to depend on the metric, since $\overline{\Box}$ does. On the other hand, Dolbeault cohomology only depends on the complex structure. So we get a connection between metric structures (like $\overline{\Box}$), and metric independent structures!

We will prove this now, using the basic properties of linear elliptic operators developed above. Before starting, we need one more tool regarding Sobolev spaces, which is the Sobolev space analog of the Arzela-Ascoli theorem.

Theorem 0.8. *(Rellich's Compactness Theorem) The inclusion $W^{k,2} \hookrightarrow W^{p,2}$ for $k > p$ is compact. That is, if $\{\varphi_j\}$ is a bounded sequence in $W^{k,2}$ then there is a convergent subsequence in $W^{p,2}$.*

(1) Convince yourself that the theory we developed for elliptic operators applies to our current setting. That is, convince yourself that the local theory applies globally by using a partition of unity. Convince yourself that $\overline{\Box}$ is a linear elliptic operator of the same form as L (say when E has rank 1)– you may assume (X, g) is Kähler, though the result is true more generally. And finally, convince yourself that everything we did for linear elliptic equations applies more generally to linear elliptic systems

$$-D_i\left(A^{ij}_{\alpha\beta}(x)D_j u^\beta\right) + b^i_{\alpha\beta}D_i u^\beta + c_{\alpha\beta}u^\beta = f_\alpha.$$

(2) Consider $\overline{\Box} : W^{2,2} \to L^2$. Let $\mathrm{Ker}\overline{\Box} = \{\varphi \in W^{2,2} : \overline{\Box}\varphi = 0\}$. Show that $\mathrm{Ker}\overline{\Box}$ is finite dimensional. (**Hint**: If not, let $\{\varphi_\ell\}$ be an infinite, orthonormal basis. Then $\|\varphi_\ell - \varphi_j\|_{W^{2,2}} = \sqrt{2}$ for $j \neq \ell$. Using elliptic regularity and Rellich's theorem, obtain a contradiction.) Note the same argument, shows $\mathrm{Ker}(\overline{\Box} - \lambda)$ is finite dimensional for all $\lambda \in \mathbb{R}$. Elliptic regularity, shows $\mathrm{Ker}(\overline{\Box} - \lambda)$ consists only of smooth sections.

(3) Prove the following "improved elliptic regularity". If $\varphi \in W^{2,2}$ and φ is orthogonal to $\mathrm{Ker}\overline{\Box}$, then

$$\|\varphi\|_{W^{2,2}} \leqslant C\|\overline{\Box}\varphi\|_{L^2}.$$

Hint: Argue by contradiction. Suppose there are φ_n orthogonal to $\mathrm{Ker}\overline{\Box}$ but such that

$$\|\varphi_n\|_{W^{2,2}} \geqslant n\|\varphi\|_{L^2}.$$

Define $\psi_n = \varphi_n\|\varphi_n\|^{-1}_{W^{2,2}}$. Use elliptic regularity and Rellich's lemma to obtain a contradiction.

(4) Define the $\mathrm{Ran}(\overline{\Box}) := \{\psi \in L^2 : \psi = \overline{\Box}\varphi \text{ for some } \varphi \in W^{2,2}\}$. Show that $\mathrm{Ran}(\overline{\Box})$ is closed. **Hint**: Use the improved regularity estimate from the last problem.

(5) Since $\mathrm{Ran}(\overline{\Box})$ is closed, we can write $L^2 = \mathrm{Ran}(\overline{\Box}) \oplus \mathrm{Ran}(\overline{\Box})^\perp$. That is, for all $\psi \in L^2$ we can write

$$\psi = \overline{\Box}\varphi + \psi_0$$

for $\varphi \in W^{2,2}$ and ψ_0 orthogonal to $\mathrm{Ran}(\overline{\Box})^\perp$. Show that there is a natural identification

$$\mathrm{Ran}(\overline{\Box})^\perp \longleftrightarrow \mathrm{Ker}(\overline{\Box}).$$

Show that this implies the orthogonal decomposition

$$L^2 = \mathrm{Ker}\overline{\Box} \oplus \mathrm{Range}\overline{\partial} \oplus \mathrm{Range}\overline{\partial}^\dagger$$

Hint: Use that $\overline{\Box}$ is self-adjoint.

(6) Finally, in the notation of the last problem, define $G : L^2 \to W^{2,2}$ by $G\psi = \varphi$. Show that

$$\overline{\Box}G\psi = (Id - \pi)\psi$$

where ψ is the orthogonal projection to $\mathrm{Ker}\overline{\Box}$. Show that

$$\|G\psi\| \leqslant C\|\psi\|$$

so $G : L^2 \to W^{2,2}$ is a bounded and continuous operator. Observe that, if we compose with the inclusion map, $W^{2,2} \hookrightarrow L^2$, then Rellich's lemma implies that $G : L^2 \to L^2$ is compact, in the sense that, if $\{\psi_j\}$ is a bounded sequence, the $\{G\psi_j\}$ contains a convergent subsequence. Show that G is self-adjoint.

(7) Prove Corollary 0.7. Deduce that if $\mathrm{Ker}\overline{\Box}_{E\otimes\Lambda^{p,q}} = 0$ then $\check{H}(X, E \otimes \Lambda^p) = H^q_{\overline{\partial}}(X, E \otimes \Lambda^p) = 0$.

(8) Let E be the trivial bundle. Note that the general strategy above works for the other Laplacians

$$\Delta_d := d^\dagger d + dd^\dagger$$

where $d : \Lambda^p \to \Lambda^{p+1}$ is the de Rham differential, as well as the ∂ Laplacian

$$\Box := \partial^\dagger\partial + \partial\partial^\dagger.$$

To convince yourself of this, just note that all we really needed to complete the above argument was that the Laplacian $\overline{\Box}$ was elliptic and self-adjoint. By definition, $\Delta_d, \Box$ share this property, and so everything goes through. As we proved in class, on a Kähler manifold, we have $\overline{\Box} = \Box = \frac{1}{2}\Delta_d$. Conclude the following famous theorem

Theorem 0.9 (Hodge Decomposition Theorem)**.** *On a compact Kähler manifold we have*

$$H^r_{dR}(M) = \bigoplus_{p+q=r} H^{p,q}(M)$$
$$H^{p,q}_{dR}(M) = \overline{H^{q,p}(M)}$$
$$H^{p,q}_{dR}(M) = H^{p,q}_{\overline{\partial}}(M) = \check{H}^(M, \Omega^p).$$

In particular, we have $h^{p,q} = \dim H^{p,q} = \dim H^{q,p} = h^{q,p}$*, and the odd Betti numbers of* M *are even.*