Math 230a: Homework 8

Due: Friday, December 1

1. Suppose (M, g) is a complete Riemannian manifold with non-positive sectional curvature. Prove that there is a map
$$\pi : \mathbb{R}^n \to M$$
such that π is smooth, surjective, and, for every point $p \in M$, there is an open set $U \subset M$, with $p \in U$ such that
$$\pi^{-1}(U) = \bigcup_\alpha V_\alpha$$
where V_α are pairwise disjoint open sets in $\mathbb{R}^n$, and $\pi : V_\alpha \to U$ is a diffeomorphism. In other words, $\mathbb{R}^n$ is the universal cover of M.

2. For a two dimensional Riemannian manifold (M, g) there is only *one* non-zero sectional curvature, which we call the *Gauss curvature*. Suppose $S \subset \mathbb{R}^3$ is a hypersurface, which we endow with a Riemannian metric by restricting the euclidean metric from $\mathbb{R}^3$. Suppose S is give near zero as the graph
$$(x, y) \mapsto (x, y, f(x, y))$$
where $f(0, 0) = 0, \nabla f(0, 0) = 0$. This can always be arranged by rotation and translation. Show that the Gauss curvature at $(0, 0)$ is proportional to $\det D^2 f(0, 0)$.

3. Suppose $S \subset \mathbb{R}^3$ is a compact surface, endowed with a metric as in the previous problem. Show that there is a point $p \in S$, where the Gauss curvature is strictly positive. Conclude, in particular, that if (S, g) is a surface with non-positive sectional curvature, then it cannot be isometrically imbedded in $\mathbb{R}^3$. (**Hint:** look at a point where S is tangent to a large sphere centered at the origin).

4. Introduce a complete Riemannian metric on $\mathbb{R}^2$. Prove that
$$\lim_{r \to \infty} \inf_{x^2 + y^2 \geq r^2} K(x, y) \leq 0$$
where $K(x, y)$ is the Gauss curvature.

5. Let M^n be an orientable Riemannian manifolds with positive sectional curvature, and even dimension. Let $\gamma(t)$ be a closed geodesic; that is $\gamma : [0, a] \to M$ with $\gamma(0) = \gamma(a)$. Show that γ is homotopic to a closed curve whose length is strictly smaller than the length of γ. Begin by showing that there is a vector $V \in T_{\gamma(0)}M$ which is orthogonal to $\gamma'(0)$ and is fixed under parallel transport around γ (**Hint:** Consider the eigenvalues of the parallel transport map, and use the orientability and the even-dimension.) Use this to define a parallel vector field along γ, and compute the second variation of the energy.