Math 230a: Homework 7

Due: Friday, November 10

1. Suppose $f_0, f_1 : M \to N$ are smooth maps, which are *homotopic* in the sense that there is a smooth map $\psi : [0,1] \times M \to N$ so that $\psi(0,\cdot) = f_0(\cdot)$ and $\psi(1,\cdot) = f_1(\cdot)$. Let ω be a closed p-form on N. Show that
$$f_1^*\omega = f_0^*\omega + d\alpha$$
for some smooth $p-1$ form α. Suppose that $U \subset M$ is a contractible set, in the sense that there is a smooth map $\psi : [0,1] \times M \to \mathbb{M}$ such that $\psi(1,\cdot)$ is the identity on U, and $\psi(0,\cdot)$ maps U to a point. Deduce that $H^p_{dR}(U) = 0$. In particular, for any p-form α with $d\alpha = 0$ there is a $p-1$ form β with $d\beta = \alpha$. If you want to, give another proof of this using ODE techniques. This is often referred to as the Poincare lemma.

2. Let $E \to M$ be a vector bundle, and let $F : N \to M$ be a map. Recall we defined the pull-back bundle $F^*E \to N$ to be the bundle obtained by pulling-back local frames from M. If ∇ is a connection on E, then we can define a pull-back connection F^*E by the rule
$$F^*\nabla_X(F^*s) = F^*\left(\nabla_{dF(X)}s\right).$$
Write down $F^*\nabla$ in local coordinates.

3. Let $E \to M$ be a vector bundle with a connection ∇. Let $p \in M$, and let $(x^1, \ldots, x^n)$ be local coordinates near p, and $\{e_1, \ldots, e_r\}$ be a local frame for E. Show that, near p we can write
$$F_\nabla = F_{ij}{}^{\alpha}{}_{\beta}\, dx^j \wedge dx^i$$
where, for each $1 \leq i, j \leq n$, $F_{ij}{}^{\alpha}{}_{\beta}$ is the endomorphism of E given by
$$s \mapsto -[\nabla_i, \nabla_j]s = \nabla_j\nabla_i - \nabla_i\nabla_j s.$$
Show that $F_{ij}{}^{\alpha}{}_{\beta}$ is indeed a section of $End(E) \otimes \bigwedge^2 T^*M$. Note that we can also view the curvature as section of $End(E) \otimes T^*M \otimes T^*M$ which has the anti-symmetry
$$F_{ij}{}^{\alpha}{}_{\beta} = -F_{ji}{}^{\alpha}{}_{\beta}.$$

4. This problem computes the first Chern class of the "holomorphic tangent bundle" over S^2– a rank 1 complex vector bundle. Recall that TS^2 can be viewed as a subbundle of the trivial $\mathbb{R}^3$ bundle over S^2– which we will denote (abusively) by $\mathbb{R}^3$– defined by
$$TS^2 = \{(x, v) : |x| = 1, \;\; x \cdot v = 0\}.$$

Equip the trivial bundle with the Euclidean metric, and let $\Pi : \mathbb{R}^3 \to TS^2$ be the orthogonal projection. The trivial bundle has a connection on it defined as follows. First, since the bundle $\mathbb{R}^3$ is trivial, a section F of $\mathbb{R}^3$ can be identified with a a triple $F = (f_1, f_2, f_3)$ where each $f_i : S^2 \to \mathbb{R}$ is a smooth function. Then we define $\nabla F = dF = (df_1, df_2, df_3) \in T^*S^2 \otimes \mathbb{R}^3$. Using the orthogonal projection Π induces a connection ∇ on TS^2 (think about how this connection relates to the Levi-Civita connection on TS^2 from the previous problem!). Now TS^2 has a complex structure given by

$$J(x)v = x \times v$$

where $\times$ denotes the usual cross product in $\mathbb{R}^3$. Show that ∇ commutes with $J(x)$ in the sense that, for any smooth vector field v we have

$$\nabla(J(x)v) = J(x)\nabla v$$

Argue that ∇ therefore descends to a connection on $T^{1,0}S^2 \subset TS^2 \otimes \mathbb{C}$, where $T^{1,0}S^2$ is the rank 1 complex vector bundle over S^2 induced by J. With this set-up, show that

$$[c_1(T^{1,0}S^2)]$$

represents the same cohomology class in $H^2_{dR}(S^2)$ as $\frac{1}{2\pi}\sqrt{detg}\, dx^1 \wedge dx^2$ of the (scaled) volume form. Here g is the Riemannian metric on S^2 induced by restriction of the Euclidean metric from $\mathbb{R}^3$. Argue that $[c_1(T^{1,0}S^2)]$ is not trivial. For this, use Stokes' Theorem. Argue that if $c_1(T^{1,0}S^2) = d\alpha$ for some 1 form α, then pulling back to $\mathbb{R}^2 = S^2 - \{(0,0,1\}$ we would have

$$\int_{S^2} \sqrt{detg} dx^1 \wedge dx^2 = \int_{S^2} d\alpha = \lim_{\epsilon \to 0} \int_{C_\epsilon} \alpha = 0$$

where C_ϵ is a circle of radius ϵ centered around $(0,0,1)$. Argue that this is impossible. In particular, conclude that there is no non-vanishing global section of $T^{1,0}S^2$.