Math 230a: Homework 6

Due: Friday, November 3

1. Let (x, y) be cartesian coordinates on $\mathbb{R}^2$, and $U = \{x_0 < x < x_1,\ y_0 < y < y_1\}$ an open rectangle in $\mathbb{R}^2$. Define $\phi : U \to \mathbb{R}^3$ by

$$\phi(x, y) = (f(y)\cos(x), f(y)\sin(x), g(y))$$

where f, g are differentiable, and $f'(y)^2 + g'(y)^2 \neq 0$, and $f(y) \neq 0$.

   (a) Show that ϕ is an immersion. The image of ϕ, called S, is a surface of revolution, and the image of the lines $x = \text{const}$ and $y = \text{const}$ are called meridians and parallels respectively.

   (b) Show that the metric on S obtained by restricting the euclidean metric from $\mathbb{R}^3$ is given in the coordinates (x, y) by

$$g = f^2 dx \otimes dx + (f')^2 + (g')^2 dy \otimes dy$$

   (c) Show that the local equations of a geodesic γ are

$$\frac{d^2x}{dt^2} + \frac{2ff'}{f^2}\frac{dx}{dt}\frac{dy}{dt} = 0$$

$$\frac{d^2y}{dt^2} - \frac{ff'}{(f')^2 + (g')^2}\left(\frac{dx}{dt}\right)^2 + \frac{f'f'' + g'g''}{(f')^2 + (g')^2}\left(\frac{dy}{dt}\right)^2 = 0$$

   (d) Show that the first equation is, except for the meridians and parallels, equivalent to the fact that $|\gamma'(t)|^2 = \text{const}$. Show that the first equation is equivalent to the fact that if $\beta(t)$ denotes the "oriented angle", $\beta < \pi$ of γ with a parallel P intersecting γ at $\gamma(t)$, then

$$r\cos\beta = \text{const}$$

   where r is the radius of the parallel P. This is called *Clairaut's relation.*

   (e) Consider the paraboloid given by $f(y) = y, g(y) = y^2$ and

$$0 < y < +\infty, -\epsilon < x < 2\pi + \epsilon.$$

   Use Clairaut's relation to show that a geodesic which is not a meridian intersects itself an infinite number of times.

2. Let $\mathcal{H}^2 := \{(x, y) \in \mathbb{R}^2 : y > 0\}$. Equip $\mathcal{H}^2$ with a metric

$$g := \frac{dx^2 + dy^2}{y^2}.$$

On homework 6 you showed that

- The lines $x = constant$ are geodesics.
- If we identify $z = x + iy$. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$, the map $z \mapsto \frac{az+b}{cz+d}$ defines an isometry of $(\mathcal{H}^2, g_2)$.
- By using the action of $SL(2, \mathbb{R})$, you showed that the geodesics of $(\mathcal{H}^2, g)$ are the lines with $x = constant$, or semi-circles with center lying on the x-axis.

(a) Show that $(\mathcal{H}^2, g)$ is complete. **Hint**: Show that the geodesics $x =$ const. exist for all time, and use the action of $SL(2, \mathbb{R})$ to conclude the same thing for all geodesics.

(b) What happens if we replace g with

$$g_\alpha = g := \frac{dx^2 + dy^2}{y^\alpha}.$$

3. A geodesic $\gamma : [0, \infty) \to (M, g)$ is called a *ray from* $\gamma(0)$ if γ minimizes the distance between $\gamma(0)$ and $\gamma(s)$ for all $s > 0$. Show that if M is complete, non-compact and $p \in M$, then M contains a ray starting from p.