Math 230a: Homework 5

Due: Friday, October 20

1. Recall that we defined a manifold M to be orientable if the tangent bundle TM was orientable as a vector bundle. Prove that the *manifold* TM is always orientable, even if M is not.

2. Prove that if E is a real vector bundle with a complex structure $J \in End(E)$, then E is orientable.

3. Prove that $\mathbb{RP}^n$ is orientable if and only if n is odd.

4. Define an operator $d : \Lambda^k T^*X \to \Lambda^{k+1} T^*X$ by the following axioms.

   - d is $\mathbb{R}$-linear.
   - for smooth functions $f \in \Lambda^0 T^*X$, df is the 1-form $df(V) = V(f)$.
   - for smooth function $d(df) = 0$.
   - for any p form α, and $k - p$ form β we have

   $$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p (\alpha \wedge d\beta).$$

   (a) Show that if $\alpha = f dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ in local coordinates, then

   $$d\alpha = \sum_j \frac{\partial f}{\partial x_j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

   Note that this formula, by linearity, specifies the action of d in general (though it's not completely obvious that this formula glues to a globally defined operator).

   (b) Show that $d(d\alpha) = 0$ for all α.

   (c) Show that, if $X_0, \ldots, X_k$ are smooth vector fields on M, then for any smooth k-form ω we have

   $$\begin{aligned} d\omega(X_0, X_1, \ldots, X_k) &= \sum_{i=0}^{k} (-1)^i X_i(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_k)) \\ &+ \sum_{0 \le i \le j \le k} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k) \end{aligned}$$

   where $\hat{X}_i$ means that we omit X_i.

(d) Show that there is a 1-form α on S^1 so that $d\alpha = 0$, but $\alpha \neq df$ for any smooth function $f : S^1 \to \mathbb{R}$. In other words

$$\frac{\text{Kernel } d : \Lambda^1 T^* S^1 \to \Lambda^2 T^* S^1}{\text{Image } d : \Lambda^0 T^* S^1 \to \Lambda^1 T^* S^1} \neq 0.$$

What if we replace S^1 with $\mathbb{R}$?

(e) More generally, show that if M is any compact manifold, and α is a 1-form so that $\alpha(p) \neq 0 \in T_p^* M$ for any $p \in M$, the $\alpha \neq df$ for any smooth function f.

5. If $\Sigma \subset \mathbb{R}^3$ is a hypersurface, we showed that Σ has a complex structure coming from the map $v \mapsto n \times v$ where n is the unit normal vector to Σ (with respect to the inner product on $\mathbb{R}^3$).

   (a) Prove that the restriction of the euclidean metric to $T\Sigma$ defines a hermitian metric on the complex vector bundle $T\Sigma_{\mathbb{C}}$.

   (b) For any manifold M, a two-form $\omega \in \Lambda^2 T^* X$ is said to be *symplectic* if ω is non-degenerate, and closed. By non-degenerate, we mean that, for all $p \in M$ if $\omega_p(X, Y) = 0$ for all $Y \in T_p M$, then $X = 0$. By closed we mean that $d\omega = 0$. If Σ is a hypersurface in $\mathbb{R}^3$, show that Σ has a symplectic form.

   (c) Show that any symplectic manifold X has dimension $2n$ (ie. even), and that X is orientable.

6. Recall the following result from differential equations

   Theorem: Let M be a manifold and X be C^∞ vector field defined on an open set $V \subset M$. For a given point $p \in V$ there exists an open set $V_0 \subset V$ and a number $\delta > 0$, and a C^∞ mapping $\phi : (-\delta, \delta) \times V_0 \to V$ with the following property: for every point $q \in V_0$, the curve $\phi(t, q)$, for $t \in (-\delta, \delta)$ is the unique trajectory of X passing through q at $t = 0$.

   Use this theorem to show that for all $p \in M$ there exists an open neighbourhood $0 \in U \subset T_p M$ so that, for all $V \in U$ there is a unique geodesic $\gamma(t)$ with initial data $\gamma(0) = p, \gamma'(0) = V$ defined for $t \in (-1, 1)$. (**Hint**: Interpret the geodesic equation as flow equation for a vector field on TM.)

7. Show that if γ is a geodesic, then $g(\gamma', \gamma')$ is constant.

8. Let (M, g), and (N, h) be two Riemannian manifolds. A diffeomorphism $f : M \to N$ is called an isometry if $f^* h = g$. Show that if f is an isometry, and γ is a geodesic in M, then $f(\gamma)$ is a geodesic in N.

9. Consider the round sphere $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ with metric induced by restricting the euclidean metric. Describe the geodesics. (**Hint**: there is a large isometry group.)

10. Let $S \subset \mathbb{R}^3$ be a surface with metric induced by restricting the euclidean metric. Show that a curve $\gamma(t)$ in $\mathbb{R}^3$ whose image lies entirely on S is a geodesic of (S, g) if and only if $\ddot{\gamma}(t)$ is orthogonal to $T_{\gamma(t)} S$ for all t.

11. Let $\mathcal{H}^2 := \{(x, y) \in \mathbb{R}^2 : y > 0\}$. Equip $\mathcal{H}^2$ with a metric

$$g_\alpha := \frac{dx^2 + dy^2}{y^2}.$$

(a) Show that the lines $x = \textit{constant}$ are geodesics. (**Hint**: use symmetry!)

(b) Identify $z = x + iy$. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$, consider the map $z \mapsto \frac{az+b}{cz+d}$. Show that this defines an isometry of $(\mathcal{H}^2, g_2)$.

(c) By using the action of $SL(2, \mathbb{R})$, show that the geodesics of $(\mathcal{H}^2, g)$ are the lines with $x = \textit{constant}$, or semi-circles with center lying on the x-axis.