Math 230a: Homework 4

Due: Friday, September 29

1. Let G be a Lie group. Show that $TG \cong G \times T_eG$ where $e \in G$ is the identity.

2. Show that $U(2)$ is diffeomorphic to $S^3 \times S^1$, and that $SU(2)$ is diffeomorphic to S^3. Conclude that TS^3 is the trivial bundle. Show the same thing for TS^1.

3. Consider $S^1 = \{e^{i\theta} \in \mathbb{C} : \theta \in [0, 2\pi)\}$. Let $\mathbb{R} \to S^1$ denote the trivial bundle, and $M \to S^1$ denote the Möbius bundle.

   (a) Consider the map $f_n : S^1 \to S^1$ given by $e^{i\theta} \mapsto e^{in\theta}$. Show that $f_n^* M$ is trivial if n is even, and non-trivial if n is odd.

   (b) Which of $\mathbb{R} \oplus \mathbb{R}$, $M \oplus \mathbb{R}$, and $M \oplus M$ are isomorphic?

   (c) Consider the inclusion $\iota : \mathbb{RP}^1 \hookrightarrow \mathbb{RP}^2$ given by

   $$[x_0 : x_1] \mapsto [x_0 : x_1 : 0].$$

   Let $\pi : S^2 \to \mathbb{RP}^1$ be the quotient map. Identify $(\iota \circ \pi)^* T\mathbb{RP}^2$ with one of the above bundles.

4. Let $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the standard almost complex structure defined by

   $$Je_i = e_{n+i}, \qquad Je_{n+i} = -e_i, \qquad 1 \le i \le n$$

   Consider the bilinear map $(x, y) \mapsto \omega(x, y) = \langle Jx, y \rangle$ mapping $\mathbb{R}^{2n} \to \mathbb{R}$. We call this the "standard symplectic form". Let $Sp(2n, \mathbb{R})$ denote the subgroup of $Gl(2n, \mathbb{R})$ preserving this bilinear form; that is, for all x, y

   $$\omega(Ax, Ay) = \omega(x, y).$$

   Show that $Sp(2n, \mathbb{R})$ is a smooth submanifold of $Gl(2n, \mathbb{R})$, compute its dimension, and identify the tangent space at the identity as a linear subspace of $M(2n, \mathbb{R})$.

5. Recall that $SO(3)$ is the group of orthogonal rotations of $\mathbb{R}^3$. If $A \in SO(3)$, then the columns of A form an oriented, orthonormal basis of $\mathbb{R}^3$.

   (a) Show that every element of $SO(3)$ fixes some line in $\mathbb{R}^3$ pointwise.

   (b) Show that $SO(3)$ is diffeomorphic to US^2 defined by

   $$US^2 := \{(p, v) : p \in S^2, \quad v \in T_pS^2, \quad |v|^2 = 1\} \subset TS^2$$

   where $|v|$ denotes the Euclidean length of v. In fact, $SO(3)$ is also diffeomorphic to $\mathbb{RP}^3$ (can you prove this?)