Math 230a: Homework 3

Due: Friday, September 22

1. Let $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$.
   (a) Describe TS^n as a subbundle of the trivial bundle $S^n \times \mathbb{R}^{n+1} \to S^n$.
   (b) Let $w = (0, 0, 1) \in \mathbb{R}^3$. For $v \in T_p S^2$ consider the map
   $$w_p^*(v) := w \cdot v$$
   where $\cdot$ denotes the usual dot product in $\mathbb{R}^3$. Show that w^* defines a global section of T^*S^2. Express w^* using stereographic projection on $S^2 - \{N\}$ and $S^2 - \{S\}$, and check that w^* satisfies the expected transition function relation on the overlap.

2. Let M be a compact manifold, and let $E \to M$ be a smooth vector bundle.
   (a) Show that M can be imbedded in $\mathbb{R}^N$ for N very large.
   (b) Show that, after possibly increasing N, we can realize $E \to M$ as a subbundle of the trivial bundle $\mathbb{R}^N \times M \to M$. (**Hint**: in both cases, use local coordinates and a partition of unity to construct a map into $\mathbb{R}^N$.)

3. Recall the manifold $\mathbb{RP}^n$ from homework 2. Define
   $$E := \{(\ell, [x]) : [x] \in \mathbb{RP}^n, \quad \ell = \lambda \cdot x \text{ for some } \lambda \in \mathbb{R}\} \subset \mathbb{R}^{n+1} \times \mathbb{RP}^n$$
   (a) Show that E is a $\mathbb{C}^\infty$ subbundle of the trivial bundle $\mathbb{R}^{n+1} \times \mathbb{RP}^n$.
   (b) Denote a point $[x] \in \mathbb{RP}^n$ by $[x_0 : x_1 : \ldots : x_n]$. Let $U_i := \{x_i \neq 0\}$. Show that (U_i, ϕ_i) defines an atlas where
   $$\phi_i([x_0 : \ldots : x_n]) = \frac{1}{x_i}(x_0, \ldots, x_n) = (w_0, \ldots, w_{i-1}, 1, w_{i+1}, \ldots, w_n) \in \mathbb{R}^n \subset \mathbb{R}^{n+1}.$$
   Show that (U_i, ϕ_i) gives a coordinate atlas on $\mathbb{RP}^n$.
   (c) Show that $\ell_i := (w_0, \ldots, w_{i-1}, 1, w_{i+1}, \ldots, w_n)$ gives a trivialization $\psi_i : E|_{U_i} \to \mathbb{R}$, and determine the transition functions $g_{ij} = \psi_i \circ \psi_j^{-1}$.
   (d) Let $p(x_0, \ldots, x_n)$ be a linear polynomial with $\mathbb{R}$ coefficients. Show that p determines a globally defined section of the dual bundle E^*, and work out the transition functions for E^*.

4. Let G be a Lie group, and let $e \in G$ be the identity. Note that G acts on itself by multiplication. For example, we have the following actions of G on itself; for $g \in G, h \in G$ consider
   $$L_g(h) = gh, \qquad R_g(h) = hg, \qquad Ad_g(h) = ghg^{-1}.$$

Each of these give smooth maps from G to itself, which are referred to as left multiplication, right multiplication and the adjoint action, respectively. Note that L_g and R_g are diffeomorphisms.

(a) Consider the Lie group $Gl(n, \mathbb{R})$. Compute the push-forward maps

$$dL_g : T_eG \to T_gG, \quad dR_g : T_eG \to T_gG, \quad dAd_g : T_eG \to T_eG.$$

Note that the formulas for dL_g and dR_g allow you to obtain the description of the tangent bundle $TG \to G$ as a subset of $M(n, \mathbb{R}) \times M(n, \mathbb{R})$.

(b) Show that for any Lie group the map

$$ad : G \to \mathrm{End}(T_eG), \qquad g \mapsto dAd_g$$

is a smooth map. Here $\mathrm{End}(T_eG)$ denotes the linear maps from T_eG to itself. Differentiating this map at e we get a map (also denoted by ad)

$$ad : T_eG \to \mathrm{End}(T_e(G)).$$

Show that there is a bilinear map $[\,,\,] : T_eG \times T_eG \to T_eG$ defined by

$$[\xi, \eta] = ad(\xi)\eta,$$

and that $[\xi, \eta] = -[\eta, \xi]$. (**Hint**: For the last statement, let $Inv : G \to G$ be the map $g \mapsto g^{-1}$. Show that $dInv_e(V) = -V$ for any $V \in T_eG$.)

(c) Specializing to the case $G = Gl(n, \mathbb{R})$, show that

$$ad(A)(B) = [A, B]$$

where $[\,,\,]$ is the matrix commutator.

(d) Let M be any Lie subgroup of $Gl(n, \mathbb{R})$. Show that $[\,,\,] : T_eM \to T_eM$ is given by the matrix commutator. (**Hint**: just note that everything commutes with restriction).