Math 230a: Homework 2

Due: Wednesday, September 13

1. (Basic properties of derivations) Let M be a manifold, and $p \in M$. Suppose S, T are derivations on C_p^∞, and $\alpha \in \mathbb{R}$. Define, for all $f \in C_p^\infty$;

   (i) $(S + T)(f) = S(f) + T(f)$

   (ii) $(\alpha T)(f) = \alpha \cdot T(f)$

   (iii) $0(f) = 0$.

   Show that this makes the set $\{D | D \text{ is a derivation on } C_p^\infty\}$ into a vector space.

2. Prove Hadamard's Lemma. If $F : \mathbb{R}^n \to \mathbb{R}$ is smooth, then there are smooth functions $H_1(x), \ldots, H_n(x)$ so that

$$F = F(0) + \sum_{i=1}^{n} x_i H_i(x)$$

   with $H_i(0) = \frac{\partial F}{\partial x_i}(0)$. (**Hint**: Use the fundamental theorem of calculus).

3. Let $p(x_1, \ldots, x_k)$ be a homogeneous polynomial of degree $m \geq 2$. That is,

$$p(tx_1, \ldots, tx_k) = t^m p(x_1, \ldots, x_k).$$

   (a) Prove that if $a \neq 0$, and $p^{-1}(a)$ is not empty, then $X_a := \{p(x) = a\}$ is a smooth, $k - 1$ dimensional submanifold of $\mathbb{R}^k$.

   (b) Prove that X_a is diffeomorphic to X_1 if $a > 0$, and X_a is diffeomorphic to X_{-1} if $a < 0$, provided a is in the range of p.

4. Let $M_n(\mathbb{R})$ be the space of $n \times n$ matrices with real entires. Assume $n \geq 2$, and define $f : M_n(\mathbb{R}) \to \mathbb{R}$ to be $f(A) = \det(A)$.

   (a) Recall that the adjoint of A has entry in the i-th row and j-th column

$$(\text{adj}A)_{ij} = (1-)^{i+j} \det A(j|i)$$

   where $A(j|i) \in M_{n-1}(\mathbb{R})$ is the matrix obtained by removing the j-th column and the i-th row. Show that the differential of f at A is given by

$$df_A : M_n(\mathbb{R}) \to \mathbb{R}, \qquad df_A(B) = \text{Tr}((\text{adj}A)B)$$

   (b) Use the fact that $A(\text{adjA}) = (\det A)I$ to recover the formula for the differential of f we used in class

$$df_A = (\det A)\text{Tr}(A^{-1}B)$$

   whenever $\det A \neq 0$.

5. Let M be a manifold, and $\sim$ be an equivalence relation. Let $\pi : M \to M/\sim$ be the quotient map sending a point $p \in M$ to it's equivalence class $[p]$. We endow $M/\sim$ with the quotient topology, determined by the following rule: $U \subset M/\sim$ is open if and only if $\pi^{-1}(U)$ is open in M.

   Let $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$ act on S^n by $x \mapsto -x$. Show that $S^n/\mathbb{Z}_2$ is has the structure of a smooth manifold. This manifold is called *real projective space* and is usually denoted $\mathbb{RP}^n$.

6. Let X, Y, Z be the vector fields on $\mathbb{R}^3$ defined by

$$X = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \quad Y = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \quad Z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

   (a) Compute the flow of the vector field X.

   (b) The map $\mathbb{R}^3 \ni (a, b, c) \mapsto aX + bY + cZ$ injects onto its image which is a subspace of the space of smooth vectorfields on $\mathbb{R}^3$. Show that, under this map, the bracket of vector fields induces the cross product on $\mathbb{R}^3$.