Math 230a: Homework 1

Due: Wednesday, September 6

1. (Stereographic projection) Let
$$S^n := \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} | x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$$
be equipped with the subset topology. That is, a set $V \subset S^n$ is open if $V = S^n \cap U$ for an open set $U \subset \mathbb{R}^{n+1}$. Let $N = (0, \dots, 0, 1)$ be the North pole, and $S = (0, \dots, 0, -1)$ be the south pole. Define $\pi_1 : S^n - \{N\} \to \mathbb{R}^n$ (resp. $\pi_2 : S^n - \{S\} \to \mathbb{R}^n$) so that $(\pi_1(p), 0)$ (resp. $(\pi_2(p), 0)$) is the point where the Line passing through N (resp. S) and p intersects the hyperplane $\{x_{n+1} = 0\}$.
   (a) Prove that $\Phi := \{(S^n - \{N\}, \pi_1), (S^n - \{S\}, \pi_2)\}$ is a C^∞ atlas on S^n.
   (b) Prove that (S^n, Φ) is a smooth submanifold on $\mathbb{R}^{n+1}$. That is, the smooth structure defined by the Φ coincides with the smooth structure induced on S^n as a submanifold of $\mathbb{R}^{n+1}$.

2. Suppose X is a connected topological space. Assume that X is Hausdorff, and locally euclidean of dimension n; that is, X can be covered by charts (U_α, ϕ_α) such that
$$\phi_\alpha : U_\alpha \to \mathbb{R}^n$$
is a homeomorphism. The following three properties are equivalent
   (a) X is second countable. That is, there is a countable collection of open sets $\{U_i\}_{i\in\mathbb{N}}$ such that, for an open set W we can write
$$W = \bigcup U_{i_k}$$
for some i_k. For example, $\mathbb{R}^n$ is second countable, where the U_i can be taken to be open balls centered on rational points, and with rational radii.
   (b) X is paracompact.
   (c) There exist compact sets $\{K_i\}_{i\in\mathbb{N}}$ such that $K_i \subset \text{int}(K_{i+1})$ and $X = \cup_i K_i$. That is, X has a compact exhaustion.

   Prove that (b) and (c) are equivalent. **Here is a "hint"**. To prove $(b) \Rightarrow (c)$, cover X by open sets which are preimages, under ϕ_α of open balls (with compact closure). By paracompactness, you can take a locally finite refinement $\{V_\alpha\}_{\alpha\in A}$ all of which have compact closure. Use these sets to construct K_i iteratively. To prove $(c) \Rightarrow (b)$, let $\{V_\alpha\}$ be any open cover. Since X is Hausdorff, compact sets are closed, and so $E_{i,j} := K_i - \text{int}(K_j)$ is compact for $j > i$. Take a finite subcover of the $\{V_\alpha\}$ covering $E_{i+1,i}$, and set
$$W_{\alpha,i} = V_\alpha \cap \text{int}(E_{i+2,i-1}).$$
Show that the resulting collection $\{W_{\alpha,i}\}$ is a locally finite refinement. For fun, prove the equivalence of $(a)/(b)$ and (c).

3. If M, N are connected, smooth manifolds, then the product $M \times N$ can be made into a smooth manifold using **the product manifold** structure. Given patches (U, ϕ) on M and (V, ψ) on N we use $(U \times V, \phi \times \psi)$ as a patch on $M \times N$. Show that this makes $M \times N$ into a smooth manifold. To show $M \times N$ is paracompact, use the preceding problem.

4. Let (x, y, z) be coordinates on $\mathbb{R}^3$. Let Y_r be the set of points in $\mathbb{R}^3$ at distance $r > 0$ from the circle
$$C = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 = 1, y = 0\}$$
   (a) Let $A = \{r \in (0, \infty) | Y_r \text{ is a submanifold of } \mathbb{R}^3\}$. Find A.
   (b) Let S^1 be equipped with the smooth structure given by stereographic projection (see (1)), and let $S^1 \times S^1$ be equipped with the product manifold structure (see below). Prove that Y_r is diffeomorphic to $S^1 \times S^1$ for any $r \in A$.