Math 136: Differential Geometry– Homework 6

Due Friday, October 20

1. Suppose that $S \subset \mathbb{R}^3$ is a parametrized surface, and let ∇ denote the covariant derivative. Let X, X_1, X_2, Y, Y_1, Y_2 be smooth vector fields on S, and f, f_1, f_2 be smooth functions on S. Prove the following statements

   (i) $\nabla_{f_1 X_1 + f_2 X_2} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y$

   (ii) $\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$

   (iii) $\nabla_X (fY) = (\nabla_X f) Y + f \nabla_X Y$

   (iv) $\nabla_X g(Y_1, Y_2) = g(\nabla_X Y_1, Y_2) + g(Y_1, \nabla_X Y_2)$

2. Prove the following lemma, mentioned in class: Let $f : U \to \mathbb{R}^3$ be a parametrized surface.

   (i) If (u^1, u^2) are local coordinates on U, then $\left[\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right] = 0$.

   (ii) Given vector fields X, Y, find a formula for $[X, Y]$.

   Hint: For part (i), recall the definition of the bracket in terms of vector fields in $\mathbb{R}^3$.

3. Show that the geodesics on the sphere are precisely the great circles.
   Hint: Identify the geodesics geometrically, and then use uniqueness to argue that these are all possible geodesics.

4. Let $S^2 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ denote the sphere, and let g denote the metric on S^2. Consider the map

$$(r, t) \mapsto (\cos r \cos t, \sin r \cos t, \sin t), \qquad (r, t) \in [0, 2\pi) \times (0, \pi) \tag{1}$$

   (i) Find the metric and the Christoffel symbols in the local coordinates given by (r, t).

   (ii) Let $p_1 = (1, 0, 0), p_2 = (0, 1, 0)$ be points in S^2. Let γ_1 be the geodesic connecting p_1 to p_2. Let V be any tangent vector in $T_{p_1} S^2$. Find the parallel transport of V along γ_1.

(iii) Let $p_3 = (0, 0, -1)$. The γ_2 be the geodesic connecting p_2 to p_3, and let γ_3 be the geodesic connecting p_3 to p_1. We define a map $A : T_{p_1}S^2 \to T_{p_1}S^2$ as follows. Given V, parallel transport V along γ_1 to get $V_2 \in T_{p_2}S^2$. Then parallel transport V_2 along γ_2 to get a vector $V_3 \in T_{p_3}S^2$. Finally parallel transport V_3 along γ_3 to get a vector, denoted $A(V)$ in $T_{p_1}S^2$. Show that A defines a linear map, and compute its matrix. This matrix is called the *holonomy*.

Hint: You only need to use your computation in part (ii), and the symmetry of the sphere

5. Find the Christoffel symbols for a surface of revolution given by

$$(u, v) \mapsto (f(v)\cos u, f(v)\sin u, g(v)), \qquad f(v) \neq 0. \tag{2}$$