Math 136: Differential Geometry– Homework 1

Due Wednesday, September 13

(1) [Kühnel, Chapter 2, Problem 1 (page 49)]. Prove that the curvature curvature and torsion of a Frenet curve $c(t)$ in $\mathbb{R}^3$ are given by the formulas

$$\kappa(t) = \frac{\|\dot{c} \times \ddot{c}\|}{\|\dot{c}\|^3}, \qquad \text{and} \qquad \tau(t) = \frac{\det(\dot{c}, \ddot{c}, \dddot{c})}{\|\dot{c} \times \ddot{c}\|^2}.$$

(2) [Kühnel, Chapter 2, Problem 4 (page 49)]. Show that a regular curve between to points $p, q \in \mathbb{R}^n$ with minimal length is necessarily the line segment from p to q. **Hint**: Consider the Schwartz inequality $\langle X, Y \rangle \leq \|X\| \cdot \|Y\|$ for the tangent vector and the difference vector $p - q$.

(3) [Kühnel, Chapter 2, Problem 8 (page 50)]. Show that the Frenet two-frame of a plane curve with given curvature functions $\kappa(s)$ can be described by the exponential series for the matrix

$$\begin{bmatrix} 0 & \int_0^s \kappa(t)dt \\ -\int_0^s \kappa(t)dt & 0 \end{bmatrix}.$$

Show that this implies that the Frenet frame is

$$\begin{bmatrix} e_1(s) \\ e_2(s) \end{bmatrix} = \sum_{i=0}^{\infty} \frac{1}{i!} \begin{bmatrix} 0 & \int_0^s \kappa(t)dt \\ -\int_0^s \kappa(t)dt & 0 \end{bmatrix}^i \cdot \begin{bmatrix} e_1(0) \\ e_2(0) \end{bmatrix}$$

Here we mean that the components of the first row of the matrix on the right hand side describe the vector $e_1(s)$ and similarly for the components of the second row of the right hand side.

(4) Let $c(s) : I \to \mathbb{R}^3$ be a Frenet curve, parametrized by arc-length. We call the vector $e_2(s) := c''/\|c''\|$ the *principal normal* vector. We say that $c(s)$ is a *Bertrand curve* if there is a function $r(s) : I \to \mathbb{R}$ such that the curve

$$\tilde{c}(s) := c(s) + r(s)e_2(s) \tag{1}$$

has $e_2(s)$ as its principal normal vector for every $s \in I$. In this case we say that $c, \tilde{c}$ are a Bertrand pair of curves. Suppose that $c(s)$ *is NOT* planar. Prove the following statements:

(i) Suppose that $c, \tilde{c}$ are a Bertrand pair of curves. Show that $r(s)$ is constant, and in particular, the distance between the curves $\|c - \tilde{c}\|$ is constant.

(ii) Show that the angle between the tangent vectors $c', \tilde{c}'$ is constant. That is, show that

$$\frac{\langle e_1(s), \tilde{c}'(s) \rangle}{\|\tilde{c}'(s)\|} = const. \tag{2}$$

Deduce that there are constants $a, b \in \mathbb{R}$ with $a > 0$ such that $aK + b\tau \equiv 1$.

(iii) Give an explicit example of a non-planar pair of Bertrand curves.