Math 112: Real Analysis– Homework 5

Due Tuesday, March 8, 2016

(1) Rudin, Chapter 2, Problems 19

(2) Rudin, Chapter 3, Problems 1, 2, 4, 5

(3) Prove that a compact metric space has a countable dense subset

OPTIONAL CHALLENGE PROBLEMS

(C1) Rudin, Chapter 3, Problems 23, 24 ,25. This sequence of problems shows how, given a metric space (X, d) one can construct another metric space $(\widehat{X}, \widehat{d})$ with the following properties

- there is a map $\iota : X \hookrightarrow \widehat{X}$– that is, we can view X as a subset of the new metric space $\widehat{X}$
- View X as a subset of $\widehat{X}$, if we restrict the metric $\widehat{d}$ to X, we get back the metric on X. That is,
$$(\iota(X), \widehat{d}|_X) = (X, d).$$
- $(\widehat{X}, \widehat{d})$ is complete.

This gives one method for constructing the real numbers from the rationals.

(C2) Let (X, d) be a metric space. For any compact subset $A \subset X$, and any $\epsilon > 0$ we set
$$B_\epsilon(A) := \bigcup_{p \in A} B_\epsilon(p).$$
This is the "ϵ- fattening" of A (draw a picture). For Y, Z compact subsets of X define the *Hausdorff distance* between Y and Z by
$$d_H(Y, Z) := \inf \{\epsilon > 0 | Y \subset B_\epsilon(Z), \quad Z \subset B_\epsilon(Y)\}.$$

(a) Show that d_H defines a metric on the set $\hat{X} := \{A \subset X | \ A \ \text{is compact}\}$.

(b) Define $\tilde{X} := \{A \subset X | \, A \text{ is compact}\}$. Show that $(\tilde{X}, d_H)$ is a compact metric space if and only if (X, d) is compact, by using the following hints:

Hint 1 It suffices to show that every infinite subset of $\tilde{X}$ contains a limit point. Let $\{A_j\}$ be an infinite set of points in $\tilde{X}$. Since $A_j \subset X$ is compact, by problem (3) above, there exists a countable dense subset

$$\{p_k^j\}_{k \in \mathbb{N}} \subset A_j.$$

These points form a sort of "skeleton" of each A_j.

Hint 2 Consider the set $E_1 := \{p_1^j\}_{j \in \mathbb{N}}$, which is an infinite subset of X. Argue that E_1 has a limit point $p_1 \in X$. In particular, find a subsequence j_ℓ so that

$$p_1^{j_\ell} \in B_{\frac{1}{\ell}}(p_1)$$

Consider the sequence of sets $\{A_{j_\ell}\}$. It suffices to show that this set has a limit point. We reindex $j_\ell \to j$ (for notational simplicity). Then we have a sequence of points $\{p_2^j\}$. Repeating the above argument we obtain a limit point p_2, and after reindexing we have

$$p_1^j \in B_{\frac{1}{j}}(p_1) \quad p_2^j \in B_{\frac{1}{j}}(p_2)$$

Continue this to find points $E_\infty := \{p_k\}_{k \in \mathbb{N}}$. Let A be the closure of E_∞. Show that, up to taking a subsequence, A_j converges to A in the metric d_H.