ANTICYCLOTOMIC EULER SYSTEMS FOR UNITARY GROUPS

ANDREW GRAHAM AND SYED WAQAR ALI SHAH

Abstract. Let $n \geq 1$ be an odd integer. We construct an anticyclotomic Euler system for certain cuspidal automorphic representations of unitary groups with signature $(1, 2n-1)$.

Contents

1. Introduction

In [Kol89], Kolyvagin constructs an *anticyclotomic* Euler system for (modular) elliptic curves over $\mathbb{Q}$ which satisfy the so-called "Heegner hypothesis". One can view the classes in this construction as images under the modular parameterisation of certain divisors on modular curves arising from the embedding of reductive groups

$$\mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_m \hookrightarrow \mathrm{GL}_2 \tag{1.1}$$

where E is an imaginary quadratic number field. These divisors (Heegner points) are defined over ring class fields of E and satisfy trace compatibility relations as one varies the field of definition. In particular, Kolyvagin shows that if the bottom Heegner point is non-torsion, then the group of E-rational points has rank equal to one. Combining this so-called "Kolyvagin step" with the Gross–Zagier formula that relates the height of this point to the derivative of the L-function at the central value, one obtains instances of the Birch–Swinnerton-Dyer conjecture in the analytic rank one case. The above construction has been generalised to higher weight modular forms and to situations where a more general hypothesis is placed on the modular form (see [Sch86], [Nek92], [BDP13] and [Zha01]).

In this paper, we consider a possible generalisation of this setting; namely, we construct an anticyclotomic Euler system for the p-adic Galois representations attached to certain regular algebraic conjugate self-dual cuspidal automorphic representations of GL_{2n}/E. More precisely, we consider the following "symmetric pair"

$$\mathrm{U}(1, n-1) \times \mathrm{U}(0, n) \hookrightarrow \mathrm{U}(1, 2n-1). \tag{1.2}$$

Note that both groups are outer forms of the groups appearing in (1.1) when $n = 1$. Let π_0 be a cuspidal automorphic representation of $\mathrm{U}(1, 2n-1)$ and let π denote a lift to the group of unitary similitudes. Under certain reasonable assumptions on π_0 (for example, non-endoscopic and tempered with stable L-packet), there exists a cuspidal automorphic representation Π of $\mathrm{GL}_{2n}(\mathbb{A}_E) \times \mathrm{GL}_1(\mathbb{A}_E)$ which is locally isomorphic to the base-change of π at all but finitely many primes. Let ρ_π denote the Galois representation attached to Π, as constructed by Chenevier and Harris [CH13].

We impose the following assumptions. We require that the lift π of π_0 as above is cohomological and the Galois-automorphic piece $\rho_\pi \otimes \pi_f$ appears in the middle degree cohomology of the Shimura variety for $\mathrm{GU}(1, 2n-1)$ – see Assumption 3.5. If this assumption is satisfied, we say that $\rho_\pi \otimes \pi_f$ admits a "modular parameterisation". Our second assumption is an analogue of the Heegner hypothesis: we require that $\pi_{0,f}$ admits a $\mathbf{H}_0$-linear model, i.e. one has

$$\mathrm{Hom}_{\mathbf{H}_0(\mathbb{A}_f)}(\pi_{0,f}, \mathbb{C}) \neq 0$$

where $\mathbf{H}_0$ is the subgroup $\mathrm{U}(1, n-1) \times \mathrm{U}(0, n)$. Finally, we require that p is split in the quadratic extension E, and π_0 (or equivalently, π) has a fixed eigenvector with respect to a certain Hecke operator associated with the Siegel parahoric level at the prime p. We refer to this assumption as "Siegel ordinarity".

We now state our main result. Let S be a finite set of primes containing all primes where π_0 is ramified, and all primes that ramify in $E/\mathbb{Q}$. For $p \notin S$, let $\mathcal{R}$ denote the set of *square-free* positive integers divisible only by primes that lie outside $S \cup \{p\}$ and split in $E/\mathbb{Q}$. For $m \in \mathcal{R}$ and an integer $r \geq 0$, we let $E[mp^r]$ denote the ring class field of E of conductor mp^r.

Theorem A. *Let $n \geq 1$ be an odd integer and $p \notin S$ an odd prime that splits in $E/\mathbb{Q}$. Let π_0 be a cuspidal automorphic representation of $\mathrm{U}(1, 2n-1)$ as above. We impose the following conditions:*

- *$\rho_\pi \otimes \pi_f$ admits a "modular parameterisation" (see Assumption 3.5).*
- *$\pi_{0,f}$ admits a $\mathbf{H}_0$-linear model.*
- *π_0 is "Siegel ordinary" (see Assumption 7.1).*

Let T_π be a Galois stable lattice inside ρ_π. Then there exists a split anticyclotomic Euler system for $T_\pi^*(1-n)$, i.e. for any collection of primes of E lying above primes in $\mathcal{R}$, and for $m \in \mathcal{R}$, $r \geq 0$, there exist classes $c_{mp^r} \in \mathrm{H}^1(E[mp^r], T_\pi^*(1-n))$ satisfying

$$\mathrm{cores}_{E[mp^r]}^{E[\ell mp^r]} c_{\ell mp^r} = \begin{cases} P_\lambda\left(\mathrm{Fr}_\lambda^{-1}\right) \cdot c_{mp^r} & \text{if } \ell \neq p \text{ and } \ell m \in \mathcal{R} \\ c_{mp^r} & \text{if } \ell = p \end{cases}$$

where λ is the prime of E lying above ℓ, $P_\lambda(X) = \det\left(1 - \mathrm{Frob}_\lambda^{-1} X | T_\pi(n)\right)$ is the characteristic polynomial of a geometric Frobenius at λ, and $\mathrm{Fr}_\lambda \in \mathrm{Gal}(E[mp^r]/E)$ denotes the arithmetic Frobenius at λ.

Remark 1.3. Evidently, we can take $c_{mp^r} = 0$ for all m, r, and the above theorem is vacuous. However, the Euler system classes we construct arise from special cycles on Shimura varieties, and the non-vanishing of these classes is expected to be related to the behaviour of the L-function attached to π_0 (more precisely, the non-vanishing of the derivative of the L-function at its central value). Proving such a relation constitutes an *explicit reciprocity law*, and we will pursue this in future work.

The above definition of a split anticyclotomic Euler system originates from the forthcoming work of Jetchev, Nekovář and Skinner, in which a general machinery for bounding Selmer groups attached to conjugate self-dual representations is developed. An interesting feature of this work is that one only needs norm relations for primes that split in the CM extension, rather than a "full" anticyclotomic Euler system. As a consequence of this, we expect to obtain the following corollary:

Corollary. *Let π_0 be as in Theorem A, and suppose that the Galois representation ρ_π satisfies the following hypotheses:*

- *The primes for which ρ_π is ramified lie above primes that split in $E/\mathbb{Q}$.*
- *The representation ρ_π is absolutely irreducible.*
- *There exists an element $\sigma \in \mathrm{Gal}\left(\overline{E}/E[1](\mu_{p^\infty})\right)$ such that*

$$\mathrm{rank}\, \rho_\pi/(\sigma - 1)\rho_\pi = 1.$$

Then, if $c := \mathrm{cores}_E^{E[1]} c_1$ is non-torsion, the Bloch–Kato Selmer group $\mathrm{H}^1_f(E, \rho_\pi(n))$ is one-dimensional.

Remark 1.4. The existence of such an element σ in the above corollary is expected to follow if the image of the Galois representation $\rho_\pi \colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_{2n}(\overline{\mathbb{Q}}_p)$ is sufficiently large. In particular, this will exclude automorphic representations of "CM type".

Remark 1.5. In this paper, we have chosen to focus on the case where E is an imaginary quadratic number field – however, we expect the results to also hold for general CM fields. Moreover, it should be possible to construct similar classes for certain inner forms of the groups appearing in (1.2). Another direction worth exploring is the case of inert primes which is closer in spirit to Kolyvagin's original construction – these questions will also be investigated further in future work.

Remark 1.6. In its simplest form the Heegner hypothesis says that the conductor of the elliptic curve is divisible only by primes that split in the imaginary quadratic extension. By the Tunnell–Saito theorem, this implies that the automorphic representation attached to the elliptic curve admits a $\mathbb{A}_{E,f}^{\times}$-linear model ([YZZ13, Theorem 1.3]). To generalise this, we start with a cuspidal automorphic representation τ_0 of $\mathrm{GL}_{2n}(\mathbb{A})$ that is of "symplectic type" (i.e. it is the automorphic base-change of a cuspidal automorphic representation σ_0 of $\mathrm{GSpin}_{2n+1}(\mathbb{A})$) and we assume that σ_0 has trivial central character. Choose an imaginary quadratic extension $E/\mathbb{Q}$ such that every prime that divides the conductor of τ_0 splits in $E/\mathbb{Q}$, and let Π_0 denote the base-change of τ_0 to $\mathrm{GL}_{2n}(\mathbb{A}_E)$, which is generically cuspidal (for example, if τ_0 is not isomorphic to its quadratic twist [AC89, Chapter 3]). Since τ_0 is of symplectic type and σ_0 has trivial central character, we have $\Pi_0 \cong \Pi_0^c \cong \Pi_0^\vee$ and Π_0 admits a local Shalika model at all primes that lie above primes that split in $E/\mathbb{Q}$. In fact, this implies that Π_0 admits a global Shalika model if we additionally assume that $\Pi_{0,\lambda}$ is a discrete series representation of $\mathrm{GL}_{2n}(E_\lambda) = \mathrm{GL}_{2n}(\mathbb{Q}_\ell)$ (see the discussion following Proposition 6.3 in [PWZ19]).

By our conditions on the ramification of τ_0, we expect that Π_0 descends to a cuspidal automorphic representation π_0 of the unitary group $\mathbf{G}_0 = \mathrm{U}(1, 2n-1)$ that is quasi-split at all finite places (with the additional assumption that n is odd), and that π_0 admits a $\mathbf{H}_0(\mathbb{A}_f)$-linear model. Indeed, for the primes where Π_0 is ramified, there is nothing to check, and for the primes where Π_0 is unramified, one should be able to check this explicitly via Satake parameters (similar to the calculations in [Sak08, §4]). If we further assume that the sign of the functional equation of the (completed) L-function of Π_0 is -1, then it is not expected that π_0 is globally distinguished, in the sense that the following period integral

$$\int_{\mathbf{H}_0(\mathbb{Q})\backslash \mathbf{H}_0(\mathbb{A})} \phi(h)dh$$

is non-vanishing, for some $\phi \in \pi_0$. If it were, then by the conjecture of Getz–Wambach (see [GW14] and [PWZ19, Remark 1.7]) this would imply that the analogous period integral for Π_0 would be non-vanishing and $L(\Pi_0, 1/2) \neq 0$ – a contradiction to our assumption on the sign of the functional equation.

1.1. **Outline of the paper.**

Let π_0 be a cuspidal automorphic representation for the unitary group $\mathbf{G}_0$ of signature $(1, 2n-1)$ (defined in section 2.1), such that $\pi_{0,\infty}$ lies in the discrete series. Discrete series L-packets of $\mathbf{G}_0$ are parameterised by irreducible algebraic representations of GL_{2n}/E; we let V_0 denote the algebraic representation corresponding to the L-packet containing $\pi_{0,\infty}$. As explained in the previous section, we assume that $\pi_{0,f}$ admits a $\mathbf{H}_0$-linear model, i.e. we have

$$\mathrm{Hom}_{\mathbf{H}_0(\mathbb{A}_f)}(\pi_{0,f}, \mathbb{C}) \neq 0.$$

In particular, this implies that V_0 is self-dual and the central character of π_0 is trivial. Let π denote a lift of π_0 to the unitary similitude group $\mathbf{G} = \mathrm{GU}(1, 2n-1)$, and let $\Pi = \omega \boxtimes \Pi_0$ denote a weak base-change to $\mathrm{GL}_1(\mathbb{A}_E) \times \mathrm{GL}_{2n}(\mathbb{A}_E)$ as explained in Proposition 3.1. We assume that Π is cuspidal and, since the central character of π_0 is trivial, we can choose π such that ω is equal to the trivial character (Lemma 2.10). Furthermore, π_∞ lies in the discrete series L-packet corresponding to the trivial extension of V_0 to an algebraic representation of $\mathrm{GL}_1/E \times \mathrm{GL}_{2n}/E$, which we denote by V (so in particular, V is also self-dual).

The representation Π decomposes as a restricted tensor product $\otimes'_v \Pi_v$ over the places of E, and for each (finite) prime λ where Π_λ is unramified, we let $L(\Pi_\lambda, s)$ denote the standard local L-factor attached to Π_λ (see Definition 2.9). We use similar notation for the representation Π_0. Combining the results of several people (see Theorem 3.4), Chenevier and Harris have shown that there exists a semisimple continuous Galois representation $\rho_\pi \colon G_E \to \mathrm{GL}_{2n}(\overline{\mathbb{Q}}_p)$ satisfying the following unramified local–global compatibility: for all but finitely many rational primes ℓ and for all $\lambda | \ell$

$$\det\left(1 - \mathrm{Frob}_\lambda^{-1}\, \ell^{-s} | \rho_\pi\right) = L(\Pi_\lambda, s + 1/2 - n)^{-1} = L(\Pi_{0,\lambda}, s + 1/2 - n)^{-1}$$

where Frob_λ is an arithmetic Frobenius at λ and the last equality follows because $\omega = 1$. We assume that this Galois representation appears in the (compactly-supported) étale cohomology of the Shimura variety

$\mathrm{Sh}_{\mathbf{G}}(K)$ associated with $\mathbf{G}$ (with appropriate level $K \subset \mathbf{G}(\mathbb{A}_f)$), or equivalently, we assume that we have a "modular parameterisation"

$$\mathrm{H}^{2n-1}_{\text{ét}}\left(\mathrm{Sh}_{\mathbf{G},\overline{\mathbb{Q}}}, \mathscr{V}(n)\right) := \varinjlim_{L} \mathrm{H}^{2n-1}_{\text{ét}}\left(\mathrm{Sh}_{\mathbf{G}}(L)_{\overline{\mathbb{Q}}}, \mathscr{V}(n)\right) \twoheadrightarrow \pi_f^{\vee} \otimes \rho_{\pi}^*(1-n).$$

where the limit is over all sufficiently small compact open subgroups $L \subset \mathbf{G}(\mathbb{A}_f)$, and $\mathscr{V}$ denotes the p-adic sheaf associated with the representation V. In analogy with the Heegner point case, the Euler system will be obtained as the images of certain classes under this modular parameterisation. We also require that π satisfies a Siegel ordinarity condition at p. Briefly, this means that the representation π_p contains a vector φ_p that is an eigenvector for the Hecke operator $\mathcal{U}_S := \mu(\tau)[J\tau J]$, where

$$\tau = 1 \times \begin{pmatrix} p & \\ & 1 \end{pmatrix} \in \mathrm{GL}_1(\mathbb{Q}_p) \times \mathrm{GL}_{2n}(\mathbb{Q}_p) = \mathbf{G}(\mathbb{Q}_p)$$

J is the parahoric subgroup associated with the Siegel parabolic subgroup (i.e. the parabolic subgroup corresponding to the partition (n,n)), and $\mu\colon \mathbf{G}(\mathbb{Q}_p) \to \mathbb{Q}_p^{\times}$ is the highest weight of V.

We consider the following extended group $\widetilde{\mathbf{G}} := \mathbf{G} \times \mathbf{T}$ where $\mathbf{T} = \mathrm{U}(1)$ is the torus controlling the variation in the anti-cyclotomic tower (see Lemma 7.6) – the group $\mathbf{H} = \mathrm{G}\left(\mathrm{U}(1, n-1) \times \mathrm{U}(0, n)\right)$ then lifts to a subgroup of $\widetilde{\mathbf{G}}$ via the map

$$(h_1, h_2) \mapsto \left(\begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix}, \frac{\det h_2}{\det h_1} \right).$$

In particular, the dimensions of the associated Shimura varieties[1] are $\dim \mathrm{Sh}_{\mathbf{H}} = n-1$ and $\dim \mathrm{Sh}_{\widetilde{\mathbf{G}}} = 2n-1$ respectively, and the inclusion $\mathbf{H} \hookrightarrow \widetilde{\mathbf{G}}$ provides us with a rich supply of codimension n cycles on the target variety, via Gysin morphisms. These cycles carry an action of both $\widetilde{\mathbf{G}}(\mathbb{A}_f)$ and $\mathrm{Gal}(\bar{E}/E)$, and via Shimura reciprocity, the Galois action can be translated into an action of $\mathbf{T}(\mathbb{A}_f)$. This will allow us to freely switch between the automorphic and Galois viewpoints.

By considering the images under the cycle class map, we obtain classes in the degree $2n$ absolute étale cohomology of $\mathrm{Sh}_{\widetilde{\mathbf{G}}}$. A consequence of the Siegel ordinarity condition is that we can modify these classes so that they are universal norms in the anticyclotomic $\mathbb{Z}_p$-extension, and since this tower is a p-adic Lie extension of positive dimension, this will force our classes to be cohomologically trivial. We then obtain classes in the first group cohomology of the étale cohomology of $\mathrm{Sh}_{\mathbf{G}}$ by applying the Abel–Jacobi map (an edge map in the Hochschild–Serre spectral sequence). The reason that these classes are universal norms will follow from the results of [Loe19], using the fact that the pair of subgroups in (1.2) give rise to a spherical variety at primes which split in the imaginary quadratic extension. This norm-compatibility will also imply the Euler system relations in the p-direction. We discuss these relations in section 6. For the convenience of the reader, we have provided the constructions of the Gysin morphisms and Abel–Jacobi maps for continuous étale cohomology in Appendix A.

The key technique introduced in [LSZ17] is to rephrase the horizontal Euler system relations as an statement in local representation theory. The underlying idea is to consider the above construction for varying levels of the source and target Shimura varieties, and then package this data into a map of $\mathbf{H}(\mathbb{A}_f^p) \times \widetilde{\mathbf{G}}(\mathbb{A}_f^p)$ representations. In section 5, we construct this map in an abstract setting using the language of cohomology functors. The ideas used here are essentially the same as those that appear in *op.cit.* but we hope that this formalism will be useful in proving Euler system relations for general pairs of reductive groups. We also note that these abstract cohomology functors appear in certain areas of representation theory, under the name "Mackey functors" (see [Dre73] for example), so these techniques can be considered as an application of Mackey theory.

Recall that $E[mp^r]$ denotes the ring class field of E corresponding to the order $\mathbb{Z} + mp^r\mathcal{O}_E$ and let $D_{mp^r} \subset \mathbf{T}(\mathbb{A}_f)$ denote the compact open subgroup corresponding to this ring class field under the Artin reciprocity map. By composing the Gysin morphisms, the Abel–Jacobi map, the modular parameterisation

[1]Strictly speaking, the group $\mathbf{H}$ does not give rise to a Shimura variety but rather what we call a *Shimura–Deligne variety* (the axiom – typically denoted (SV3) – is not satisfied). In practice this does not affect the arguments in the paper, so the reader can safely pretend that it gives rise to a Shimura variety in the usual sense. We provide justifications for this viewpoint in Appendix B.

above and passing to the "completion" (see Proposition 5.11) for varying levels, we construct a collection of maps

$$\mathcal{H}\left(\widetilde{\mathbf{G}}(\mathbb{A}_f)\right)^{1\times D_{mp^r}} \xrightarrow{\mathcal{L}_{mp^r}} \pi_f^\vee \otimes \mathrm{H}^1\left(E[mp^r], \rho_\pi^*(1-n)\right)$$

which satisfy a certain equivariance property under the action of $\widetilde{\mathbf{G}}(\mathbb{A}_f) \times \mathbf{H}(\mathbb{A}_f)$. Here $\mathcal{H}(-)^C$ denotes the elements of the associated Hecke algebra with rational coefficients which are invariant under right-translation by elements in C (and the convolution product is with respect to the fixed Haar measure in §2.1).

Let φ be an element of π_f which is fixed by the Siegel parahoric level at p. We define our Euler system to be

$$c_{mp^r} \approx \left(v_\varphi \circ \mathcal{L}_{mp^r}\right)(\phi^{(m)})$$

for suitable test data $\phi^{(m)}$, where v_φ denotes the evaluation map at the the vector φ. Here, by approximately we mean up to some suitable volume factors which ensure that the classes c_{mp^r} land in the cohomology of a Galois stable lattice. To prove the tame norm relation at $\ell \nmid m$, we choose a character $\chi\colon \mathrm{Gal}(E[mp^r]/E) \to \mathbb{C}^\times$, which can naturally be viewed as a character of $\mathbf{T}(\mathbb{A}_f)$ via the Artin reciprocity map, and consider the χ-component $\mathcal{L}^\chi_{mp^r}$ of the map $\mathcal{L}_{mp^r}$. It is enough to prove the tame norm relation locally at ℓ and, by applying Frobenius reciprocity, the map can naturally be viewed as an element

$$\left(\mathcal{L}^\chi_{mp^r}\right)_\ell \in \mathrm{Hom}_{\mathbf{H}_0(\mathbb{Q}_\ell)}(\pi_{0,\ell}, \chi^{-1} \boxtimes \chi).$$

This reduces the tame norm relation to a calculation in local representation theory, which constitutes the main part of this paper. Once this relation is established, we then sum over all χ-components to obtain the full norm relation.

We now describe the ingredients that go into the local computation described above, all of which take place in section 4 of the paper. If ℓ is a rational prime that splits in $E/\mathbb{Q}$, then there exists a prime $\lambda|\ell$ and an isomorphism $\pi_{0,\ell} \cong \Pi_{0,\lambda}$ compatible with the identification $\mathbf{G}_0(\mathbb{Q}_\ell) \cong \mathrm{GL}_{2n}(\mathbb{Q}_\ell)$. Furthermore, since $\pi_{0,f}$ admits a $\mathbf{H}_0$-linear model, we also have

$$\mathrm{Hom}_{\mathbf{H}_0(\mathbb{Q}_\ell)}(\pi_{0,\ell}, \mathbb{C}) \neq 0.$$

We show (Proposition 4.3) that this implies that $\pi_{0,\ell}$ has a local Shalika model and as a consequence of this, we prove the following theorem:

Theorem 1.7. *Let $\chi\colon \mathbb{Q}_\ell^\times \to \mathbb{C}^\times$ be an unramified finite-order character. Then*

(1) $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbf{H}_0(\mathbb{Q}_\ell)}(\pi_{0,\ell}, \chi^{-1} \boxtimes \chi) = 1$, where $\chi^{-1}\boxtimes\chi$ denotes the character of $\mathbf{H}_0(\mathbb{Q}_\ell)$ given by sending a pair of matrices (h_1, h_2) to $\chi(\det h_2/\det h_1)$.

(2) There exists a compactly-supported locally constant function $\phi\colon \mathbf{G}_0(\mathbb{Q}_\ell) \to \mathbb{Z}$ (independent of χ) such that, for all $\mathfrak{z} \in \mathrm{Hom}_{\mathbf{H}_0(\mathbb{Q}_\ell)}(\pi_{0,\ell}, \chi^{-1} \boxtimes \chi)$ and all spherical vectors $\varphi_0 \in \pi_{0,\ell}$

$$\mathfrak{z}(\phi \cdot \varphi_0) = \frac{\ell^{n^2}}{\ell - 1} L(\pi_{0,\ell} \otimes \chi, 1/2)^{-1} \mathfrak{z}(\varphi_0). \tag{1.8}$$

The first part of this theorem is a result of Jacquet and Rallis [JR96] in the case that χ is trivial, and a result of Chen and Sun [CS20] in the general case. Since $\pi_{0,\ell}$ admits a local Shalika model, one obtains an explicit basis of the hom-space by considering an associated zeta integral, and the proof of part 2 then involves manipulating this zeta integral by applying several U_ℓ-operators to its input. By taking a suitable linear combination of these operators, we are able to simultaneously produce the relation appearing in (1.8) without losing the eventual integrality of the Euler system classes. Such a linear combination arises naturally from an inclusion-exclusion principle on flag varieties associated with GL_m.

1.2. **Notation and conventions.** We fix the following notation and conventions throughout the paper:

- An embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and an imaginary quadratic number field $E \subset \overline{\mathbb{Q}}$.
- For any rational prime ℓ, we fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$ and an isomorphism

$$\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$$

  compatible with the fixed embeddings $\left(\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}, \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell\right)$. In particular, for each ℓ, we have a choice of λ above ℓ.
- A rational prime $p > 2$ that splits in E.

- $\mathbb{A}_F$ will denote the adeles of a number field F and $\mathbb{A}_{F,f}$ the finite adeles; if $F = \mathbb{Q}$ we simply write $\mathbb{A}$ and $\mathbb{A}_f$ respectively. For any integer (or finite set of primes) S, we write $\mathbb{A}^S$ (resp. $\mathbb{A}_f^S$) to mean the adeles (resp. finite adeles) away from S (i.e. at all primes not dividing S/not in S), and $\mathbb{A}_S$ or $\mathbb{Z}_S$ for the adeles at S.
- For a ring R and a connected reductive group $\mathbf{G}$ over $\mathbb{Q}$, we let
$$\mathcal{H}\left(\mathbf{G}(\mathbb{A}_f), R\right) := \left\{ \phi \colon \mathbf{G}(\mathbb{A}_f) \to R : \begin{array}{c} \phi \text{ is locally constant} \\ \text{and compactly-supported} \end{array} \right\}$$
denote the Hecke algebra, which carries an action of $\mathbf{G}(\mathbb{A}_f)$ given by right-translation of the argument. We will omit the ring R from the notation when it is clear from the context. Additionally, if $K \subset \mathbf{G}(\mathbb{A}_f)$ is a compact open subgroup, then $\mathcal{H}\left(K \backslash \mathbf{G}(\mathbb{A}_f)/K\right)$ will denote the subset of K-biinvariant functions. We also have similar notation for the $\mathbb{Q}_\ell$-points of $\mathbf{G}$.
- For a number field F, we let G_F denote the absolute Galois group of F. Furthermore, the global Artin reciprocity map
$$\mathrm{Art}_F \colon \mathbb{A}_F^\times \to G_F^{\mathrm{ab}}$$
is defined geometrically, i.e. it takes a uniformiser to the associated geometric Frobenius.
- For a prime v of F, we let Frob_v denote an arithmetic Frobenius in G_F. If L/F is an abelian extension of number fields that is unramified at v, we will sometimes write Fr_v for the arithmetic Frobenius in $\mathrm{Gal}(L/F)$ associated with the prime v.
- For an imaginary quadratic number field E and an integer m, we let $E[m]$ denote the ring class field of conductor m, i.e. the finite abelian extension with norm subgroup given by $E^\times \cdot \widehat{\mathcal{O}}_m^\times$ where
$$\mathcal{O}_m = \mathbb{Z} + m\mathcal{O}_E.$$
- The smooth dual of a representation will be denoted by $(-)^\vee$ and the linear dual by $(-)^*$.
- For a smooth quasi-projective scheme X over a characteristic zero field, and a number field F, we let $\mathrm{CHM}(X)_F$ denote the category of relative Chow motives over X with an F-structure, as defined in [Tor19, Definition 2.4]. Let $\mathrm{DM}_{B,c}(X)$ denote the category of *constructible Beilinson motives*, as in [CD19, Definition 15.1.1]. Then, for any Chow motive $(Y, e, n) \in \mathrm{CHM}(X)_\mathbb{Q}$, one can associate a constructible Beilinson motive $eM_X(Y)(n)$ as the mapping fibre
$$eM_X(Y)(n) := \mathrm{Cone}\left(M_X(Y)(n) \xrightarrow{1-e} M_X(Y)(n)\right)[-1]$$
where $M_X(Y)$ is as in [CD19, §11.1.2]. This association assembles into a functor $\mathrm{CHM}(X)_\mathbb{Q} \to \mathrm{DM}_{B,c}(X)$, so we can naturally view a Chow motive as a constructible Beilinson motive. If (Y, e, n) is equipped with an F-structure, then so is $eM_X(Y)(n)$ (i.e. one has a $\mathbb{Q}$-linear homomorphism $F \hookrightarrow \mathrm{End}_{\mathrm{DM}_{B,c}(X)}\left(eM_X(Y)(n)\right)$).
- The motivic cohomology of X with values in a relative Chow motive $\mathscr{F} \in \mathrm{CHM}(X)_F$ is the F-vector space given by
$$\mathrm{H}^\bullet_{\mathrm{mot}}\left(X, \mathscr{F}\right) := \mathrm{Hom}_{\mathrm{DM}_{B,c}(X)}\left(\mathbf{1}_X, \mathscr{F}[\bullet]\right)$$
where $\mathbf{1}_X$ denotes the trivial motive. This definition is compatible with the one arising from algebraic K-theory by [CD19, Corollary 14.2.14], and since the categories $\mathrm{DM}_{B,c}(-)$ satisfy the six functor formalism, we can define pushforwards/pullbacks between these cohomology groups (see [Lem17, §4.1] and the references therein).
- All étale cohomology groups refer to continuous étale cohomology in the sense of Jannsen ([Jan88]). A subscript c will denote the compactly supported version.
- If H and G are locally profinite groups, and σ, π smooth representations of H and G respectively, then $\sigma \boxtimes \pi$ will denote the tensor product representation of $H \times G$.
- Unless specified otherwise, all automorphic representations are assumed to be unitary.
- Unless specified otherwise, all modules over a group ring will be *left* modules.
- Let ℓ be a rational prime. For a positive integer n we write $[n]_\ell := (\ell^n - 1)(\ell - 1)^{-1}$. This definition extends to $[n]_\ell! := [n]_\ell \cdot [n-1]_\ell \cdots [1]_\ell$ and
$$\begin{bmatrix} n \\ m \end{bmatrix}_\ell := \frac{[n]_\ell!}{[m]_\ell! \cdot [(n-m)]_\ell!}$$
with the obvious convention that $[0]_\ell! = 1$.

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2. Preliminaries

2.1. **The groups.** In this section, we will define the algebraic groups and their associated Shimura data that will appear throughout the paper. All groups and morphisms will be defined over $\mathbb{Q}$ unless specified otherwise, and maps of algebraic groups will be defined on their R-points for $\mathbb{Q}$-algebras R. For an arbitrary algebraic group $\mathbf{K}$, we denote by $\mathbf{T_K}$ its maximal abelian quotient, and $\mathbf{Z_K}$ its centre.

Let $J_{r,s}$ denote the $(r+s)\times(r+s)$ diagonal matrix $\mathrm{diag}(1,\dots,1,-1,\dots,-1)$ comprising of r copies of 1 and s copies of -1. We define $\mathrm{U}(p,q)$ (resp. $\mathrm{GU}(p,q)$) to be the unitary (resp. unitary similitude) groups whose R-points for a $\mathbb{Q}$-algebra R are given by

$$\begin{aligned} \mathrm{U}(p,q)(R) &:= \left\{g \in \mathrm{GL}_{p+q}(E\otimes_{\mathbb{Q}} R) \colon {}^t\bar{g}J_{p,q}g = J_{p,q}\right\} \\ \mathrm{GU}(p,q)(R) &:= \left\{g \in \mathrm{GL}_{p+q,E}(E\otimes_{\mathbb{Q}} R) \colon \text{ there exists } \lambda \in \mathbb{G}_m(R) \text{ s.t. } {}^t\bar{g}J_{p,q}g = \lambda J_{p,q}\right\}. \end{aligned}$$

The group $\mathrm{GU}(p,q)$ comes with two morphisms

$$c\colon \mathrm{GU}(p,q)\to\mathbb{G}_m \qquad \det\colon \mathrm{GU}(p,q)\to \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$$

where $c(g)=\lambda$ is the similitude factor and $\det(g)$ is the determinant. To ease notation, we will omit q if it is zero.

Remark 2.1. These groups arise as the generic fibre of group schemes over $\mathbb{Z}$ by replacing E with $\mathcal{O}_E$ in the definition. If ℓ is unramified in E, the pullback to $\mathbb{Q}_\ell$ is unramified and the $\mathbb{Z}_\ell$-points form a hyperspecial maximal compact subgroup.

Let $n\geq 1$ be an integer. We define:

- $\mathbf{G}_0 := \mathrm{U}(1,2n-1)$.
- $\mathbf{G} := \mathrm{GU}(1,2n-1)$.
- $\mathbf{H}_0 := \mathrm{U}(1,n-1)\times\mathrm{U}(0,n)$.
- $\mathbf{H} := \mathrm{GU}(1,n-1)\times_c\mathrm{GU}(0,n)$ where the product is fibred over the similitude map.

We have natural inclusions $\iota\colon \mathbf{H}_0\hookrightarrow\mathbf{G}_0$ and $\iota\colon\mathbf{H}\hookrightarrow\mathbf{G}$, both given by sending an element (h_1,h_2) to the $2n\times 2n$ block matrix

$$\iota(h_1,h_2) = \begin{pmatrix} h_1 & \\ & h_2\end{pmatrix}.$$

Throughout, we will let $\mathbf{T}$ denote the algebraic torus $\mathrm{U}(1)=\{x\in\mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_{m,E} : \bar{x}x=1\}$ and we note that

- $\mathscr{N}\colon \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_{m,E}\to\mathbf{T}$ is given by $z\mapsto\overline{z}/z$.
- $\mathbf{Z}_{\mathbf{G}_0}=\mathbf{T}$, $\mathbf{Z}_{\mathbf{H}_0}=\mathbf{T}\times\mathbf{T}$
- $\mathbf{Z_G}=\mathrm{GU}(1)$, $\mathbf{Z_H}\cong\mathbf{T}\times\mathrm{GU}(1)$
- $\mathbf{T_H}\cong\mathbf{T}\times\mathrm{GU}(1)$ if n is odd, and $\mathbf{T_H}\cong\mathbf{T}\times\mathbf{T}\times\mathbb{G}_m$ if n is even.
- $\mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_{m,E}\cong\mathrm{GU}(1)$.

When n is odd, the quotient map is given explicitly by

$$\mathbf{H}\to\mathbf{T_H},\qquad (h_1,h_2)\mapsto(\det h_2/\det h_1, c(h_1)^{(n+1)/2}\det(h_1^{-1})).$$

Remark 2.2. This quotient in fact contains more information than we are interested in – to control the anticyclotomic variation of our Euler system we only need the fact that $\mathbf{T}$ is a quotient of $\mathbf{H}$.

We consider the group $\widetilde{\mathbf{G}} = \mathbf{G} \times \mathbf{T}$. The group $\mathbf{H}$ extends to a subgroup of $\widetilde{\mathbf{G}}$ via the map $(h_1, h_2) \mapsto (\iota(h_1, h_2), \nu(h_1, h_2))$, where $\nu(h_1, h_2) = \det h_2 / \det h_1$. We will need the following two lemmas later on:

Lemma 2.3. *The map*

$$\begin{array}{rcl} \mathscr{N} : \mathbb{A}_{E,f}^{\times} & \to & \mathbf{T}(\mathbb{A}_f) \\ z & \mapsto & \bar{z}/z \end{array}$$

is open and surjective.

Proof. A continuous surjective homomorphism of σ-countable locally compact groups is open, and so it suffices to prove that the map is surjective. We have an exact sequence $1 \to \mathbb{G}_m \to \mathrm{GU}(1) \xrightarrow{\mathscr{N}} \mathbf{T} \to 1$ of algebraic tori over $\mathbb{Q}$, which gives an exact sequence

$$1 \to \mathbb{A}_f^{\times} \to \mathbb{A}_{E,f}^{\times} \to \mathbf{T}(\mathbb{A}_f).$$

For each finite place ℓ, the map $(E \otimes \mathbb{Q}_\ell)^\times \to \mathbf{T}(\mathbb{Q}_\ell)$ is surjective, since $\mathrm{H}^1(\mathbb{Q}_\ell, \mathbb{G}_m) = 0$ by Hilbert's theorem 90. For $\ell \nmid \mathrm{disc}(E)$, both $\mathrm{GU}(1)$ and $\mathbf{T}$ have smooth models over $\mathbb{Z}_\ell$ and the map extends to these models. This induces a map $\mathrm{GU}(1)(\mathbb{F}_\ell) \to \mathbf{T}(\mathbb{F}_\ell)$. Since $\mathrm{H}^1(\mathbb{F}_\ell, \mathbb{G}_m) = 0$ by Lang's lemma, this map is surjective and a lifting argument shows that $\mathrm{GU}(1)(\mathbb{Z}_\ell) \to \mathbf{T}(\mathbb{Z}_\ell)$ is also surjective. The claim for surjectivity on adelic points now follows. □

Lemma 2.4. *Consider the following groups*

$$Z_0 := Z_{\mathbf{G}_0}(\mathbb{A}) \quad \textit{and} \quad Z_1 := Z_{\mathbf{G}}(\mathbb{A})\mathbf{G}(\mathbb{Q}) \cap \mathbf{G}_0(\mathbb{A})$$

Then $Z_1 = Z_0 \mathbf{G}_0(\mathbb{Q})$.

Proof. The inclusion $Z_0 \mathbf{G}_0(\mathbb{Q}) \subset Z_1$ is obvious. Let $z\gamma \in Z_1$ where $z \in Z_{\mathbf{G}}(\mathbb{A})$ and $\gamma \in \mathbf{G}(\mathbb{Q})$. Then we have

$$c(z\gamma) = z\bar{z}c(\gamma) = 1$$

which implies that $c(\gamma)$ is a norm locally everywhere. By the Hasse norm theorem, we must have that $c(\gamma)$ is a global norm, i.e. there exists an element $\zeta \in E^\times = Z_{\mathbf{G}}(\mathbb{Q})$ such that $c(\gamma) = c(\zeta)$. Therefore, we can replace z with $z\zeta \in Z_0$ and γ with $\zeta^{-1}\gamma \in \mathbf{G}_0(\mathbb{Q})$. □

Notation. We fix left Haar measures dg, dh and dt on $\mathbf{G}(\mathbb{Q}_\ell)$, $\mathbf{H}(\mathbb{Q}_\ell)$ and $\mathbf{T}(\mathbb{Q}_\ell)$ respectively. As these groups are unimodular [Ren10, p. 58], these are also right Haar measures. We normalise them so that the volume of any compact open subgroup lies in $\mathbb{Q}$ and the volume of any hyperspecial subgroup is 1 (when applicable). In particular, the products of these measures induce Haar measures on the $\mathbb{A}_f$-points of the corresponding groups, which we may also denote by the same letters.

Remark 2.5. If n is odd then the above groups are quasi-split at all finite places. Indeed, the groups $\mathbf{H}_0$ and $\mathbf{H}$ are automatically quasi-split at all finite places because there is only one unitary group (up to isomorphism) attached to Hermitian spaces of fixed odd dimension over a non-archimedean local field (see [Mor10, p. 152]). For $\mathbf{G}_0$ and $\mathbf{G}$, we follow the argument in *loc.cit.*, where the local cohomological invariant is calculated (the group is quasi-split when the invariant is 0).

We note that these groups are automatically quasi-split at primes ℓ which do not divide the discriminant of E, because the group is unramified. If ℓ divides the discriminant of E, then the cohomological invariant is 0 if -1 is a norm in $\mathbb{Q}_\ell$. Otherwise, it is equal to $2n - 1 + n = 3n - 1$ modulo 2. But this quantity is even, under the assumption that n is odd.

Quasi-splitness at all finite places is used in the results invoked on weak base-change in section 3. However, we expect such results to hold without this hypothesis.

2.2. **The Shimura–Deligne data.** We now define the Shimura–Deligne data associated with $\mathbf{H}$ and $\mathbf{G}$. Since the datum attached to the group $\mathbf{H}$ doesn't satisfy all the usual axioms of a Shimura datum, we have extended the definition to that of a Shimura–Deligne datum (Appendix B). In this section, we will freely use notation and results from this appendix.

Let $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_m$ denote the Deligne torus and let $h_{\mathbf{G}}$ and $h_{\mathbf{H}}$ be the following algebraic homomorphisms:

$$\begin{aligned} h_{\mathbf{G}}\colon \mathbb{S} &\to \mathbf{G}_{\mathbb{R}} \\ z &\mapsto (\mathrm{diag}(z, \bar{z}, \ldots, \bar{z})) \\ h_{\mathbf{H}}\colon \mathbb{S} &\to \mathbf{H}_{\mathbb{R}} \\ z &\mapsto (\mathrm{diag}(z, \bar{z}, \ldots, \bar{z}), \mathrm{diag}(\bar{z}, \ldots, \bar{z}))\,. \end{aligned}$$

Let $X_{\mathbf{G}}$ (resp. $X_{\mathbf{H}}$) denote the $\mathbf{G}(\mathbb{R})$ (resp. $\mathbf{H}(\mathbb{R})$) conjugacy class of homomorphisms containing $h_{\mathbf{G}}$ (resp. $h_{\mathbf{H}}$). Then the pairs $(\mathbf{G}, X_{\mathbf{G}})$ and $(\mathbf{H}, X_{\mathbf{H}})$ both satisfy the axioms of a Shimura–Deligne datum as in Definition B.1 (axioms [Del79, 2.1.1.1-2.1.1.2]). The datum $(\mathbf{G}, X_{\mathbf{G}})$ also satisfies 2.1.1.3 in *loc.cit.* so it is a Shimura datum in the usual sense. In particular, if $K \subset \mathbf{G}(\mathbb{A}_f)$ is a sufficiently small compact open subgroup (Definition B.5), we let $\mathrm{Sh}_{\mathbf{G}}(K)$ denote the associated Shimura–Deligne variety whose complex points equal

$$\mathrm{Sh}_{\mathbf{G}}(K)(\mathbb{C}) = \mathbf{G}(\mathbb{Q})\backslash[X_{\mathbf{G}} \times \mathbf{G}(\mathbb{A}_f)/K]$$

and similarly for $\mathbf{H}$. The reflex field for the datum $(\mathbf{G}, X_{\mathbf{G}})$ is the imaginary quadratic number field E fixed in the previous section for $n > 1$, and $\mathbb{Q}$ otherwise (see Example B.1). However, we will always consider the associated Shimura–Deligne varieties as schemes over E, unless stated otherwise. If the level K is small enough, the embedding ι induces a closed embedding on the associated Shimura–Deligne varieties. More precisely

Definition 2.6. Let w equal the $2n \times 2n$ block diagonal matrix

$$w := \iota(1, -1) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}.$$

We say that a compact open subgroup $K \subset \mathbf{G}(\mathbb{A}_f)$ is $\mathbf{H}$-*small* if there exists a compact open subgroup $K' \subset \mathbf{G}(\mathbb{A}_f)$, containing both K and wKw, such that for any $g \in \mathbf{G}(\mathbb{A}_f)$, $gK'g^{-1} \cap \mathbf{G}(\mathbb{Q})$ has no non-trivial stabilisers for its action on $X_{\mathbf{G}}$.

Proposition 2.7. *Let $K \subset \mathbf{G}(\mathbb{A}_f)$ be an $\mathbf{H}$-small compact open subgroup. Then the natural map*

$$\mathrm{Sh}_{\mathbf{H}}(K \cap \mathbf{H}(\mathbb{A}_f)) \xrightarrow{\iota} \mathrm{Sh}_{\mathbf{G}}(K)$$

is a closed immersion.

Proof. This follows the same argument in [LSZ17, Proposition 5.3.1]. Indeed, one has an injective map $\iota\colon \mathrm{Sh}_{\mathbf{H}} \hookrightarrow \mathrm{Sh}_{\mathbf{G}}$ on the infinite level Shimura–Deligne varieties. So it suffices to prove that for any $u \in K - K \cap \mathbf{H}(\mathbb{A}_f)$, one has $\mathrm{Sh}_{\mathbf{H}}\, u \cap \mathrm{Sh}_{\mathbf{H}} = \varnothing$ as subsets of $\mathrm{Sh}_{\mathbf{G}}$. If Q and Qu are both in $\mathrm{Sh}_{\mathbf{H}}$, we must have that $u' = u \cdot wu^{-1}w$ fixes Q. By the conditions on K', we have that $u' = 1$. This implies that u is in the centraliser of w, which is precisely $\mathbf{H}(\mathbb{A}_f)$. □

We also consider the trivial Shimura–Deligne datum $(\mathbf{T}, X_{\mathbf{T}})$, where $X_{\mathbf{T}}$ is the singleton $h_{\mathbf{T}}(z) = \bar{z}/z$ (considered as a $\mathbf{T}(\mathbb{R})$-conjugacy class of homomorphisms $\mathbb{S} \to \mathbf{T}$) and the datum $(\widetilde{\mathbf{G}}, X_{\widetilde{\mathbf{G}}}) = (\widetilde{\mathbf{G}}, X_{\mathbf{G}} \times X_{\mathbf{T}})$. Then the data associated with the groups $\mathbf{H}$ and $\widetilde{\mathbf{G}}$ are clearly compatible with respect to the inclusion (ι, ν), and if $U = K \times C \subset \widetilde{\mathbf{G}}(\mathbb{A}_f)$ is a sufficiently small compact open subgroup, we have a finite unramified map

$$\mathrm{Sh}_{\mathbf{H}}\left(U \cap \mathbf{H}(\mathbb{A}_f)\right) \to \mathrm{Sh}_{\widetilde{\mathbf{G}}}\left(U\right) \cong \mathrm{Sh}_{\mathbf{G}}(K) \times \mathrm{Sh}_{\mathbf{T}}(C).$$

which is a closed embedding when K is $\mathbf{H}$-small. The reflex field for $(\widetilde{\mathbf{G}}, X_{\widetilde{\mathbf{G}}})$ is equal to the imaginary quadratic field E.

2.3. **Dual groups, base-change, and L-factors.** In this section, we describe the dual groups attached to the groups defined in the previous section and several maps between them. Let W_F denote the Weil group of a local or global field F. In each of the following examples, we describe the connected component of the L-group. The Weil group W_E will always act trivially on this component, so to describe the full L-group, it will be enough to specify how complex conjugation acts.

As in [Ski12, §2.2], let $\Phi_m = \left((-1)^{i+1}\delta_{i,m+1-j}\right)_{i,j}$. The dual groups are as follows:

- $\widehat{\mathbf{G}} = \mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_{2n}(\mathbb{C})$ and complex conjugation acts by
$$(\lambda; g) \mapsto (\lambda \cdot \det g; \Phi_{2n}^{-1t} g^{-1} \Phi_{2n}).$$
- $\widehat{\mathbf{H}} = \mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ and complex conjugation acts as
$$(\lambda; h_1, h_2) \mapsto (\lambda \cdot \det h_1 \cdot \det h_2; \Phi_n^{-1t} h_1^{-1} \Phi_n, \Phi_n^{-1t} h_2^{-1} \Phi_n)$$
- $\widehat{\mathbf{G}}_0 = \mathrm{GL}_{2n}(\mathbb{C})$ and $\widehat{\mathbf{H}}_0 = \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ and complex conjugation acts via the same formulae above, but omitting the GL_1 factor.
- $\widehat{\mathbf{T}} = \mathrm{GL}_1(\mathbb{C})$ and complex conjugation acts by sending an element x to its inverse x^{-1}.
- $\widehat{\widetilde{\mathbf{G}}} = \widehat{\mathbf{G}} \times \widehat{\mathbf{T}}$ and complex conjugation acts diagonally.

We also note that we can choose a splitting such that the dual group of $R\,\mathrm{GL}_m := \mathrm{Res}_{E/\mathbb{Q}} \mathrm{GL}_m$ is equal to $\mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C})$ and complex conjugation acts by $(g_1, g_2) \mapsto \left(\Phi_m^{-1t} g_2^{-1} \Phi_m, \Phi_m^{-1t} g_1^{-1} \Phi_m\right)$. This implies that the natural diagonal embedding $\widehat{\mathbf{G}}_0 \to \widehat{R\,\mathrm{GL}_{2n}}$ extends to a map of L-groups, which we will denote by BC. Similarly, we have a map $\mathrm{BC}\colon {}^L\mathbf{G} \to {}^L(R\,\mathrm{GL}_1 \times R\,\mathrm{GL}_{2n})$. We also have a natural map ${}^L\mathbf{G} \to {}^L\mathbf{G}_0$ given by projecting to the second component, and this is compatible with base-change in the sense that we have a commutative diagram:

$$\begin{array}{ccc} {}^L\mathbf{G} & \xrightarrow{\mathrm{BC}} & {}^L(R\,\mathrm{GL}_1 \times R\,\mathrm{GL}_{2n}) \\ \downarrow & & \downarrow \\ {}^L\mathbf{G}_0 & \xrightarrow{\mathrm{BC}} & {}^L R\,\mathrm{GL}_{2n} \end{array}$$

where the right-hand vertical arrow is projection to the second component.

The map ${}^L\mathbf{G} \to {}^L\mathbf{G}_0$ corresponds on the automorphic side to restricting an automorphic representation of $\mathbf{G}$ to $\mathbf{G}_0$. It turns out that all irreducible constituents in this restriction lie in the same L-packet, and one can always lift a representation of $\mathbf{G}_0$ to one of $\mathbf{G}$. We summarise this in the following proposition.

Proposition 2.8. *Let π be a cuspidal automorphic representation of $\mathbf{G}(\mathbb{A})$. Then all irreducible constituents of $\pi|_{\mathbf{G}_0}$ are cuspidal and lie in the same L-packet with L-parameter obtained by post-composing the L-parameter for π with the natural map ${}^L\mathbf{G} \to {}^L\mathbf{G}_0$. Conversely, if π_0 is a cuspidal automorphic representation of $\mathbf{G}_0(\mathbb{A})$, then π_0 lifts to a cuspidal automorphic representation of $\mathbf{G}(\mathbb{A})$.*

Moreover, the same statements hold if we work over a local field F, that is to say: if π is an irreducible admissible representation of $\mathbf{G}(F)$, then all irreducible constituents of $\pi|_{\mathbf{G}_0}$ lie in the same local L-packet (with parameter obtained by composing with ${}^L\mathbf{G}_F \to {}^L\mathbf{G}_{0,F}$), with cuspidal representations corresponding to each other, and every representation of $\mathbf{G}_0(F)$ lifts to one of $\mathbf{G}(F)$. If $F = \mathbb{R}$ and π_0 lies in the discrete series, then we can lift π_0 to a discrete series representation of $\mathbf{G}(\mathbb{R})$.

If ω_{π_0} denotes the central character of a representation π_0, and ω is a (unitary) character extending ω_{π_0} to $Z_{\mathbf{G}}(\mathbb{Q})\backslash Z_{\mathbf{G}}(\mathbb{A})$, then there exists a lift π of π_0 as above with central character ω. A similar statement holds for local fields.

Proof. See Proposition 1.8.1 (and the discussion preceeding it) in [HL04] and Theorem 1.1.1 in [LS18]. The last part follows from the techniques used in [LS18, Theorem 5.2.2] and Lemma 2.4. □

We will frequently need to talk about local L-factors attached to unramified representations of the groups defined in §2.1, so for completeness, we recall the definition here. We follow §1.11, §1.14 and §5.1 Example (b) in [BR94].

Definition 2.9. Let ℓ be a prime which is unramified in $E/\mathbb{Q}$ and for which $\mathbf{G}_{0,\mathbb{Q}_\ell}$ is quasi-split. For a positive integer m, let Std_m denote the standard representation of GL_m.

(a) Let F be a finite extension of $\mathbb{Q}_\ell$. For $\mathscr{G} \in \{\mathrm{GL}_{2n}, \mathrm{GL}_1 \times \mathrm{GL}_{2n}\}$, let $r_{\mathrm{std}}\colon {}^L\mathscr{G}_F \to \mathrm{GL}_{2n}(\mathbb{C})$ denote the following representation:
$$r_{\mathrm{std}}(g, x) = \begin{cases} \mathrm{Std}_{2n}(g) & \text{if } \mathscr{G} = \mathrm{GL}_{2n} \text{ with } {}^L\mathscr{G}_F = \mathrm{GL}_{2n}(\mathbb{C}) \times W_F \\ (\mathrm{Std}_1 \boxtimes \mathrm{Std}_{2n})(g) & \text{if } \mathscr{G} = \mathrm{GL}_1 \times \mathrm{GL}_{2n} \text{ with } {}^L\mathscr{G}_F = \mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_{2n}(\mathbb{C}) \times W_F \end{cases}$$
Let q denote the order of the residue field of F and Frob an arithmetic Frobenius in W_F. For any unramified representation σ of $\mathscr{G}(F)$, with corresponding (unramified) Langlands parameter

$\varphi_\sigma \colon W_F \to {}^L\mathscr{G}_F$, the *standard* local L-factor is defined to be

$$L(\sigma, s) := \det\left(1 - q^{-s}(r_{\mathrm{std}} \circ \varphi_\sigma)(\mathrm{Frob}^{-1})\right)^{-1}$$

where $s \in \mathbb{C}$.

(b) If ℓ splits in $E/\mathbb{Q}$, then we have an isomorphism $\mathbf{G}_{0,\mathbb{Q}_\ell} \cong \mathrm{GL}_{2n,\mathbb{Q}_\ell}$, and for any unramified representation σ of $\mathbf{G}_0(F)$, we define the local L-factor of σ to be as in part (a), viewing σ as a representation of $\mathrm{GL}_{2n}(F) \cong \mathbf{G}_0(F)$. We have a similar definition for representations of $\mathbf{G}(F)$.

2.4. **Discrete series representations.** One has an identification $\mathbf{G}_E = \mathrm{GL}_{1,E} \times \mathrm{GL}_{2n,E}$ given by sending an element g to $(c(g); g')$, where g' is the matrix obtained by projecting its entries to the first component of the identification

$$\begin{array}{rcl} E \otimes_{\mathbb{Q}} E & \xrightarrow{\sim} & E \oplus E \\ x \otimes \lambda & \mapsto & (\lambda x, \bar{\lambda} x). \end{array}$$

We let $\mathbb{G}_m^{1+2n} \subset \mathbf{G}_E$ denote the standard torus with respect to this identification. Then any algebraic character of this torus is of the form

$$\begin{array}{rcl} \mathbb{G}_m^{1+2n} & \to & \mathbb{G}_m \\ (t_0; t_1, \dots, t_{2n}) & \mapsto & t_0^{c_0} \cdot \prod_i t_i^{c_i} \end{array}$$

where $\mathbf{c} = (c_0; c_1, \dots, c_{2n}) \in \mathbb{Z}^{1+2n}$. Such a character is called *dominant* if $c_1 \geq \cdots \geq c_{2n}$ and these characters classify all irreducible algebraic representations of $\mathbf{G}_E$. Indeed, any such representation corresponds to a representation with highest weight given by $\mathbf{c}$. There is a similar description for $\mathbf{G}_0$ by omitting the first $\mathbb{G}_m$-factor.

The L-packets of discrete series representations of $\mathbf{G}(\mathbb{R})$ are described by the above algebraic representations. More precisely, following [Clo91, §3.3] (the extension to similitude groups is immediate), if π_∞ lies in the discrete series, the Langlands parameter $\varphi_\infty \colon W_{\mathbb{R}} \to {}^L\mathbf{G}$ associated with π_∞, restricted to $W_{\mathbb{C}} = \mathbb{C}^\times$, is equivalent to a parameter of the form

$$z \mapsto \left((z/\bar{z})^{p_0}; (z/\bar{z})^{p_1 + \frac{2n-1}{2}}, \dots, (z/\bar{z})^{p_{2n} + \frac{2n-1}{2}}\right)$$

with $p_i \in \mathbb{Z}$ and $p_1 > \cdots > p_{2n}$. Then π_∞ corresponds to an irreducible algebraic representation of $\mathbf{G}_E$ with highest weight $\mathbf{c} = (c_0; c_1, \dots, c_{2n})$ satisfying $c_i = p_i + (i-1)$. We have a similar description for discrete series L-packets for $\mathbf{G}_0(\mathbb{R})$.

Lemma 2.10. *Let π_0 be a cuspidal automorphic representation of $\mathbf{G}_0(\mathbb{A})$ such that $\pi_{0,\infty}$ lies in the discrete series L-packet corresponding to the highest weight $\mathbf{c} = (c_1, \dots, c_{2n})$. Suppose that the central character of π_0 is trivial. Then there exists a cuspidal automorphic representation π of $\mathbf{G}(\mathbb{A})$ such that:*

- *π has trivial central character.*
- *π_∞ lies in the discrete series L-packet corresponding to the weight $(0; c_1, \dots, c_{2n})$.*

Proof. By looking at the infinitesimal character of $\pi_{0,\infty}$, we must have $c_1 + \cdots + c_{2n} = 0$ because the central character is trivial. By Proposition 2.8, there exists a cuspidal automorphic representation π of $\mathbf{G}(\mathbb{A})$ lifting π_0 such that π_∞ lies in the discrete series. Furthermore, we can arrange it so that the central character of π is trivial. In particular, if $(c_0; c_1, \dots, c_{2n})$ is the weight corresponding to the L-packet of π_∞, then since π has trivial central character, we must have

$$c_0 + \sum_{i=1}^{2n} c_i = 0.$$

This implies that $c_0 = 0$. □

We call π in the above lemma a *unitary lift* of π_0. We are interested in algebraic representations V of $\mathbf{G}_E$ with highest weight $\mathbf{c} = (0; c_1, \dots, c_{2n})$ that are self dual: in this case, we must have $\mathbf{c} = -\mathbf{c}'$, where $\mathbf{c}' = (0; c_{2n}, \dots, c_1)$. We have the following "branching law":

Lemma 2.11. *Let V be a self-dual algebraic representation of $\mathbf{G}_E$ with highest weight $\mathbf{c} = (0; c_1, \dots, c_{2n})$ as above and let $\iota^* V$ denote its restriction to an algebraic representation of $\mathbf{H}_E$. Then $\iota^* V$ contains the trivial representation as a direct summand, with multiplicity one. The same statement holds for representations of $\widetilde{\mathbf{G}}_E$ of the form $\widetilde{V} := V \boxtimes \mathbf{1}$, with respect to the embedding (ι, ν).*

Proof. This is well-known and follows from Proposition 6.3.1 in [GRG14] for example. □

These self-dual representations will correspond to the coefficient sheaves that we will consider throughout the paper. More precisely we make the following definition.

Definition 2.12. Let $\mathbf{c}$ be a dominant weight of $\mathbf{G}_E$ satisfying $\mathbf{c} = -\mathbf{c}'$ (so the associated representation $V = V_{\mathbf{c}}$ is self dual). We consider the following algebraic representation of $\widetilde{\mathbf{G}}_E$ given by $\widetilde{V} := V \boxtimes \mathbf{1}$. The restriction $(\iota, \nu)^\star \widetilde{V}$ contains the trivial representation of $\mathbf{H}_E$ with multiplicity one, by Lemma 2.11. We denote the inclusion of this factor by br.

2.5. **Coefficient sheaves.** Let p be a prime that splits in $E/\mathbb{Q}$ and let $\mathfrak{P}$ denote the prime of E lying above p which is fixed by the embedding $E \hookrightarrow \overline{\mathbb{Q}}_p$ (so $E_{\mathfrak{P}}$ is identified with $\mathbb{Q}_p$). In this section we briefly describe how to associate motivic and $\mathfrak{P}$-adic sheaves to algebraic representations of $\mathbf{G}$. We follow [Pin92], [Anc15], [Tor19] and [LSZ17] closely (although note that the conventions are slightly different in the latter to that of the first three – we follow conventions in the first three references). In particular, this will mean that the multiplier character of $\mathbf{G}$ is sent to Tate motive under the functor $\mathrm{Anc}_{\mathbf{G}}$. Since the group $\mathbf{H}$ does not give rise to a Shimura datum in the usual sense, we have provided justifications for the results in this section in Appendix C.

Let T_n be a finite-dimensional representation of a (sufficiently small) compact open subgroup $K \subset \mathbf{G}(\mathbb{A}_f)$ with coefficients in $\mathbb{Z}/p^n\mathbb{Z}$. Since T_n is finite, there exists a finite-index normal subgroup $L \subset K$ that acts trivially on T_n. We let $\mu_{\mathbf{G},K,n}(T_n)$ be the locally constant étale sheaf of abelian groups on $\mathrm{Sh}_{\mathbf{G}}(K)$ corresponding to

$$(\mathrm{Sh}_{\mathbf{G}}(L) \times T_n) / \Gamma \to \mathrm{Sh}_{\mathbf{G}}(K)$$

where $\Gamma := K/L$ acts via $(x, t) \cdot h = (xh, h^{-1}t)$. This construction is independent of the choice of L.

If $x_0 = \mathrm{Spec}(k)$ is a point on $\mathrm{Sh}_{\mathbf{G}}(K)$ and $x = \mathrm{Spec}(\bar{k})$ is the associated geometric point obtained from fixing a separable closure of k, then we can explicitly describe the action of $\mathrm{Gal}(\bar{k}/k)$ on the stalk $(\mu_{\mathbf{G},K,n}(T_n))_x$. Indeed, lift x to a geometric point $\widetilde{x}$ on $\mathrm{Sh}_{\mathbf{G}}(L)$. Then for $\sigma \in \mathrm{Gal}(\bar{k}/k)$, there exists a unique element $\psi(\sigma) \in \Gamma$ such that

$$\sigma \cdot \widetilde{x} = \widetilde{x}^{\psi(\sigma)}.$$

Since the Galois action commutes with that of Γ, the map ψ defines a homomorphism $\psi \colon \mathrm{Gal}(\bar{k}/k) \to \Gamma$. The stalk $(\mu_{\mathbf{G},K,n}(T_n))_x$ is isomorphic to T_n with the Galois action given by ψ.

The functors $\mu_{\mathbf{G},K,n}$ are compatible as n varies: if we let T be a continuous representation of K with coefficients in $\mathbb{Z}_p$, then we let $\mathscr{T}_K = \mu_{\mathbf{G},K}(T)$ denote the corresponding sheaf on $\mathrm{Sh}_{\mathbf{G}}(K)$. By considering the category of lisse sheaves on $\mathrm{Sh}_{\mathbf{G}}(K)$ with coefficients in $\mathbb{Z}_p$ up to isogeny, this construction extends to finite-dimensional continuous representations of $\mathbf{G}(\mathbb{Q}_p)$ with coefficients in $\mathbb{Q}_p$. In particular, any algebraic representation of $\mathbf{G}_{\mathbb{Q}_p}$ gives rise to a continuous representation of $\mathbf{G}(\mathbb{Q}_p)$ (with coefficients in $\mathbb{Q}_p$). One obtains an additive tensor functor

$$\mu_{\mathbf{G},K} \colon \mathrm{Rep}_{\mathbb{Q}_p}(\mathbf{G}) \to \text{Ét}(\mathrm{Sh}_{\mathbf{G}}(K))_{\mathbb{Q}_p}$$

from the category of algebraic representations of $\mathbf{G}$ over $\mathbb{Q}_p$ to the category of lisse sheaves on $\mathrm{Sh}_{\mathbf{G}}(K)$ with coefficients in $\mathbb{Q}_p$.

In [Anc15] (see also [Tor19, §10]), Ancona shows that this functor $\mu_{\mathbf{G},K}$ fits into a commutative diagram (up to natural isomorphism)

$$\begin{array}{ccc} \mathrm{Rep}_E(\mathbf{G}) & \xrightarrow{\mathrm{Anc}_{\mathbf{G},K}} & \mathrm{CHM}\left(\mathrm{Sh}_{\mathbf{G}}(K)\right)_E \\ {\scriptstyle -\otimes E_{\mathfrak{P}}}\downarrow & & \downarrow{\scriptstyle r_{\text{ét}}} \\ \mathrm{Rep}_{\mathbb{Q}_p}(\mathbf{G}) & \xrightarrow{\mu_{\mathbf{G},K}} & \text{Ét}\left(\mathrm{Sh}_{\mathbf{G}}(K)\right)_{\mathbb{Q}_p} \end{array}$$

where $\mathrm{CHM}(-)_E$ denotes the category of relative Chow motives (over E) and $r_{\text{ét}}$ denotes the $\mathfrak{P}$-adic realisation of a motive. The functor $\mathrm{Anc}_{\mathbf{G},K}$ is additive and preserves duals and tensor products.

Remark 2.13. Ancona's construction applies to general PEL Shimura–Deligne varieties (see Appendix C). In particular, since $\mathrm{Sh}_{\mathbf{H}}$ is also PEL, all of the constructions in this section hold for the group $\mathbf{H}$. Furthermore, these constructions extend to representations of $\widetilde{\mathbf{G}}$ of the form $V \boxtimes \mathbf{1}$ (note that $\widetilde{\mathbf{G}}$ does not give rise to a PEL datum).

2.6. **Equivariant coefficient sheaves.** If we wish to study the sheaves $\mathscr{V}_K = \mu_{\mathbf{G},K}(V)$ and $\mathrm{Anc}_{\mathbf{G},K}(V)$ as K varies, we need to keep track of the action of $\mathbf{G}(\mathbb{A}_f)$. More precisely, let $\text{Ét}\left(\mathrm{Sh}_{\mathbf{G}}\right)_{\mathbb{Q}_p}$ (resp. $\mathrm{CHM}\left(\mathrm{Sh}_{\mathbf{G}}\right)_E$) denote the category of equivariant étale sheaves (resp. relative Chow motives) – that is to say – a collection of sheaves $\mathscr{V}_K$ and isomorphisms $\sigma^*\mathscr{V}_K \cong \mathscr{V}_L$ for any $\sigma \in \mathbf{G}(\mathbb{A}_f)$ such that $\sigma^{-1}L\sigma \subset K$.

By [LSZ17, Proposition 6.2.4], the functors $\mu_{\mathbf{G},K}$ and $\mathrm{Anc}_{\mathbf{G},K}$ are compatible as K varies, so give rise to functors:

- $\mathrm{Anc}_{\mathbf{G}} \colon \mathrm{Rep}_E(\mathbf{G}) \to \mathrm{CHM}\left(\mathrm{Sh}_{\mathbf{G}}\right)_E$
- $\mu_{\mathbf{G}} \colon \mathrm{Rep}_{\mathbb{Q}_p}(\mathbf{G}) \to \text{Ét}\left(\mathrm{Sh}_{\mathbf{G}}\right)_{\mathbb{Q}_p}$

which are compatible under $\mathfrak{P}$-adic realisations. Given such an equivariant sheaf $\mathscr{V}$, one can define

$$\mathrm{H}^\bullet\left(\mathrm{Sh}_{\mathbf{G}}, \mathscr{V}\right) := \varinjlim_K \mathrm{H}^\bullet\left(\mathrm{Sh}_{\mathbf{G}}(K), \mathscr{V}_K\right)$$

which is a smooth admissible $\mathbf{G}(\mathbb{A}_f)$-representation. We will sometimes drop the level from the subscript when we talk about the cohomology of these equivariant sheaves at finite level.

Remark 2.14. As in the previous section, the above results also apply to the group $\mathbf{H}$ and representations of $\widetilde{\mathbf{G}}$ of the form $V \boxtimes \mathbf{1}$.

Let $\mathrm{Rep}_E(\widetilde{\mathbf{G}})_0$ denote the full subcategory of $\mathrm{Rep}_E(\widetilde{\mathbf{G}})$ consisting of representations of the form $V \boxtimes \mathbf{1}$. We have the following compatibility with base-change.

Proposition 2.15. *One has a commutative diagram (up to natural isomorphism):*

$$\begin{array}{ccc} \mathrm{Rep}_E(\widetilde{\mathbf{G}})_0 & \xrightarrow{\mathrm{Anc}_{\widetilde{\mathbf{G}}}} & \mathrm{CHM}\left(\mathrm{Sh}_{\widetilde{\mathbf{G}}}\right)_E \\ {\scriptstyle (\iota,\nu)^*}\downarrow & & \downarrow{\scriptstyle (\iota,\nu)^*} \\ \mathrm{Rep}_E(\mathbf{H}) & \xrightarrow{\mathrm{Anc}_{\mathbf{H}}} & \mathrm{CHM}\left(\mathrm{Sh}_{\mathbf{H}}\right)_E \end{array}$$

Proof. By [Tor19, Theorem 9.7] the theorem holds for the pair of groups $(\mathbf{G}, \mathbf{H})$ since the morphism of associated PEL data is "admissible" (Definition 9.1 in *op.cit.*). The result immediately extends to the category $\mathrm{Rep}_E(\widetilde{\mathbf{G}})_0$. □

We will apply this proposition to the representations $\widetilde{V}$ defined in Definition 2.12. Indeed, since $(\iota,\nu)^\star\widetilde{V}$ contains the trivial representation with multiplicity one, we have a map of relative Chow motives

$$\mathrm{br} \colon E \to (\iota,\nu)^\star \,\mathrm{Anc}_{\widetilde{\mathbf{G}}}(\widetilde{V}).$$

This "branching law" will allow us to construct classes in the motivic cohomology of $\mathrm{Sh}_{\widetilde{\mathbf{G}}}$ with coefficients in $\mathrm{Anc}_{\widetilde{\mathbf{G}}}(\widetilde{V}) \otimes E(n)$, where $E(-1)$ denotes the Lefschetz motive.

3. Cohomology of unitary Shimura varieties

In this section we recall the construction of Galois representations attached to cuspidal automorphic representations for the group $\mathbf{G} = \mathrm{GU}(1, 2n-1)$. To apply the results of [Mor10], we assume that n is *odd* throughout this section.

Let π_0 be a cuspidal automorphic representation of $\mathbf{G}_0(\mathbb{A})$ such that $\pi_{0,\infty}$ lies in the discrete series, and let π denote a lift to $\mathbf{G}(\mathbb{A})$ (see Proposition 2.8). If π_0 has trivial central character, then we take π to be a lift with trivial central character. Assume that π is cohomological, i.e. there exists:

- A sufficiently small level $K \subset \mathbf{G}(\mathbb{A}_f)$ such that $\pi_f^K \neq \{0\}$.
- An irreducible algebraic representation V of $\mathbf{G}_E$ and an integer $i \in \mathbb{Z}$ such that

$$\mathrm{H}^i\left(\mathfrak{g}_{\mathbb{C}}, K'_\infty; \pi_\infty \otimes V_{\mathbb{C}}\right) \neq \{0\}$$

where $\mathfrak{g} = \mathrm{Lie}(\mathbf{G}(\mathbb{R}))$ and $K'_\infty = K_\infty Z_{\mathbf{G}}(\mathbb{R})^\circ$, with K_∞ a maximal compact subgroup of $\mathbf{G}(\mathbb{R})$.

We say that a representation $\Pi = \psi \boxtimes \Pi_0$ of $\mathrm{GL}_1(\mathbb{A}_E) \times \mathrm{GL}_{2n}(\mathbb{A}_E)$ is θ-stable if $\Pi_0^c \cong \Pi_0^\vee$ and $\psi = \psi^c \omega_{\Pi_0}^c$, where ω_{Π_0} is the central character of Π_0. Recall that our choice of Hermitian pairing implies that the group $\mathbf{G}$ is quasi-split at all finite places. Then we have the following result due to Morel:

Proposition 3.1. *Let S be the set of places of $\mathbb{Q}$ containing ∞, all primes which ramify in E and all places where K is not hyperspecial (so π is unramified outside S).*

(1) There exists an irreducible admissible representation Π of $\mathrm{Res}_{E/\mathbb{Q}} \mathbf{G}_E(\mathbb{A})$ such that for all but finitely many primes

$$\varphi_{\Pi_\ell} \cong \mathrm{BC}\left(\varphi_{\pi_\ell}\right) \tag{3.2}$$

where φ_{Π_ℓ} and φ_{π_ℓ} are the local Langlands parameters of Π_ℓ and π_ℓ respectively, and BC is the local base-change morphism defined in section 2.3. We say Π is a weak base-change of π.

(2) If the representation Π above is cuspidal then any weak base-change of π is a θ-stable regular algebraic cuspidal automorphic representation and (3.2) holds for all $\ell \notin S$. Furthermore, the infinitesimal character of Π_∞ is the same as that of $(V \otimes V^\theta)^*$, where V^θ is the representation with highest weight $(c_0 + \cdots + c_{2n}; -c_{2n}, \ldots, -c_1)$.

(3) Suppose that Π is cuspidal and π has trivial central character (for example, if $\pi_{0,f}$ admits a $\mathbf{H}_0$-linear model). Then under the isomorphism $\mathrm{Res}_{E/\mathbb{Q}} \mathbf{G}_E(\mathbb{A}_\mathbb{Q}) \cong \mathrm{GL}_1(\mathbb{A}_E) \times \mathrm{GL}_{2n}(\mathbb{A}_E)$, the representation Π is of the form $\mathbf{1} \boxtimes \Pi_0$ for some conjugate self-dual regular algebraic cuspidal automorphic representation Π_0 of $\mathrm{GL}_{2n}(\mathbb{A}_E)$ satisfying $\varphi_{\Pi_{0,\ell}} = \mathrm{BC}\left(\varphi_{\pi_{0,\ell}}\right)$ for all $\ell \notin S$.

Proof. Most of the proposition follows from Corollary 8.5.3, Remark 8.5.4 and the proof of Lemma 8.5.6 in [Mor10]. The reason that (3.2) holds for all $\ell \notin S$ follows from the fact that $\mathbf{G}$ is quasi-split at all finite places. Moreover, if $\pi_{0,f}$ admits a $\mathbf{H}_0$-linear model then the assertion that Π is of the form $\mathbf{1} \boxtimes \Pi_0$ follows from the commutative diagram in section 2.3 and the fact that the central character of π_f is trivial. Indeed, if we write $\Pi = \psi \boxtimes \Pi_0$ then it is enough to show that $\psi_\ell = 1$ for all primes $\ell \notin S$, and this follows from a direct computation involving Satake parameters. For the assertion about the infinitesimal character see the proofs of [Mor10, Lemma 8.5.6] and [Ski12, Theorem 9]. ☐

Remark 3.3. If the highest weight of the representation V is regular then the above proposition follows from Theorem A in [Ski12] (which also builds on the work of Morel). However we are only interested in the case where Π is cuspidal which does not require this assumption.

From now on we assume that any weak base change of π is *cuspidal*. In this case, thanks to the work of many people (see for example [CH13], [HT01], [Mor10], [Shi11] and [Ski12]), there exists a semisimple Galois representation attached to π.

Theorem 3.4. *Let π be as above and assume that the central character of π is trivial. Suppose that a weak base-change $\Pi = \mathbf{1} \boxtimes \Pi_0$ of π is cuspidal. Then there exists a continuous semisimple Galois representation*

$$\rho_\pi \colon G_E \to \mathrm{GL}(W_\pi) \cong \mathrm{GL}_{2n}(\overline{\mathbb{Q}}_p)$$

such that

(1) ρ_π is conjugate self-dual (up to a twist), i.e. one has an isomorphism

$$\rho_\pi^c \cong \rho_\pi^*(1-2n)$$

where ρ_π^c denotes the representation given by conjugating the argument by complex conjugation in $G_\mathbb{Q}$ (recall that we have fixed an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$).

(2) For all $\ell \notin S \cup \{p\}$ and $\lambda | \ell$, the representation $\rho_\pi|_{E_\lambda}$ is unramified and satisfies

$$P_\lambda(\mathrm{Nm}\,\lambda^{-s}) := \det\left(1 - \mathrm{Frob}_\lambda^{-1} (\mathrm{Nm}\,\lambda)^{-s} |\, \rho_\pi\right) = L\left(\Pi_\lambda, s + (1-2n)/2\right)^{-1}.$$

In particular, if ℓ splits in $E/\mathbb{Q}$ and λ is the prime lying above ℓ corresponding to the fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$, then $P_\lambda(\ell^{-n}) = L(\pi_\ell, 1/2)^{-1}$.

(3) If π_p is unramified (i.e. $p \notin S$) then for any prime v dividing p the representation $\rho_v := \rho_\pi|_{E_v}$ is crystalline with jumps in the Hodge–Tate filtration occurring at

$$\begin{cases} c_i + 2n - i & i = 1, \ldots, 2n \quad \text{if } v \text{ is the place fixed by the embedding } \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p \\ -c_{2n+1-i} + 2n - i & i = 1, \ldots, 2n \quad \text{otherwise} \end{cases}$$

Furthermore, via the fixed isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_p$, the characteristic polynomial of the crystalline Frobenius φ on $\mathbf{D}_{\mathrm{cris}}(\rho_v)$ is equal to the characteristic polynomial of $\varphi_{\Pi_{0,v}\otimes|\cdot|_v^{1/2-n}}(\mathrm{Frob}_v^{-1})$, where

$$\varphi_{\Pi_{0,v}\otimes|\cdot|_v^{1/2-n}} : W_{E_v} \to \mathrm{GL}_{2n}(\mathbb{C})$$

is the local Langlands parameter attached to the representation $\Pi_{0,v} \otimes |\cdot|_v^{1/2-n}$.

Proof. See Theorem 3.2.3 in [CH13]. The third part follows from the explicit recipe of (1.5) in *op.cit.*. Note that the representation ρ_π, which a priori is associated with Π, only depends on π. This is because the Galois representation constructed by Chenevier–Harris is semisimple (so is determined by its behaviour at all but finitely many primes, for any finite set of primes), and Π is a weak-base change of π (so its local Langlands parameters are determined by π for all but finitely many primes). □

In [Mor10, Corollary 8.4.6], Morel shows that, up to multiplicity, this representation appears in the intersection cohomology of the Baily–Borel compactification of $\mathrm{Sh}_{\mathbf{G}}$. However, to be able to construct an Euler system, we would like this representation to occur in the (compactly-supported) cohomology of the open Shimura variety. In fact to obtain the correct Euler factor, we work with the dual of the representation and assume that it appears as a quotient of the (usual) étale cohomology.

We would also like ρ_π to be absolutely irreducible (which is expected because we have assumed that Π is cuspidal) – however this is currently not known in general. We summarise this in the following assumption:

Assumption 3.5. *Let π_0 be a cuspidal automorphic representation of $\mathbf{G}_0(\mathbb{A})$ such that $\pi_{0,\infty}$ lies in the discrete series, and let π be a lift to $\mathbf{G}(\mathbb{A})$. Suppose that π is cohomological with respect to the representation V, and suppose that any weak base-change of π is cuspidal. Then we assume that:*

- *ρ_π is absolutely irreducible.*
- *There exists a $G_E \times \mathbf{G}(\mathbb{A}_f)$-equivariant surjective map*

$$\mathrm{pr}_{\pi^\vee} : \mathrm{H}^{2n-1}_{\text{ét}}\left(\mathrm{Sh}_{\mathbf{G},\overline{\mathbb{Q}}}, \mathscr{V}^*(n)\right) \otimes \overline{\mathbb{Q}}_p \twoheadrightarrow \pi_f^\vee \otimes W_\pi^*(1-n).$$

(Here, we are identifying $\overline{\mathbb{Q}}_p$ with $\mathbb{C}$ via the fixed isomorphism in section 1.2).

Remark 3.6. If π is tempered, non-endoscopic and the global L-packet containing π is stable, then we expect that the above assumptions hold true for the following reasons. Since π is tempered the cohomology should vanish outside the middle degree and π is not a CAP representation.[2] In particular, the latter should imply that π_f does not contribute to the cohomology of the boundary strata of the Baily–Borel compactification of $\mathrm{Sh}_{\mathbf{G}}$. Since the L-packet containing π is stable, this will imply that any weak base-change of π is cuspidal, and finally, since π is non-endoscopic then (at least for p large enough) the representation ρ_π should be absolutely irreducible of dimension $2n$.

Remark 3.7. If we assume that $\Pi_{0,v}$ is supercuspidal at some non-archimedean place v of E, then the Galois representation ρ_π will be absolutely irreducible for local reasons. In fact, by choosing p large enough, we can even ensure that ρ_π is residually absolutely irreducible (see [LTX+19, Lemma E.8.3]).

4. Horizontal norm relations

This section comprises of the local calculations needed to prove the "tame norm relations" for our anticyclotomic Euler system at split primes. In [LSZ17], the key input for proving the norm relations is the existence of a Bessel model and a zeta integral that computes the local L-factor of the automorphic representation. The analogous notion in our case will be that of a Shalika model. This section is entirely local and independent of the rest of the paper – in particular, we can relax the assumption that n is odd.

For this section only, we consider

$$G = \mathrm{GL}_1 \times \mathrm{GL}_{2n} \qquad H = \mathrm{GL}_1 \times \mathrm{GL}_n \times \mathrm{GL}_n$$
$$G_0 = \mathrm{GL}_{2n} \qquad H_0 = \mathrm{GL}_n \times \mathrm{GL}_n$$

which we view as algebraic groups over $\mathbb{Z}_\ell$ and come with diagonal embeddings $H \hookrightarrow G$ and $H_0 \hookrightarrow G_0$. We fix an irreducible, smooth, admissible, unramified representation σ of $G_0(\mathbb{Q}_\ell)$ and let $\pi = \mathbf{1} \boxtimes \sigma$ be its

[2] CAP = cuspidal but associated to a parabolic.

extension to $G(\mathbb{Q}_\ell)$. For our applications later on, it will be necessary to phrase everything in terms of the extended group $\widetilde{G} := G \times \mathrm{GL}_1$. The group H lifts to a subgroup of $\widetilde{G}$ via the map

$$(\alpha, h_1, h_2) \mapsto \left(\alpha, \begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix}, \nu(h_1, h_2)\right)$$

where $\nu(h_1, h_2) = \det h_2 / \det h_1$. In particular, for any character $\chi\colon \mathbb{Q}_\ell^\times \to \mathbb{C}^\times$, the product $\pi \boxtimes \chi^{-1}$ is a representation of $\widetilde{G}(\mathbb{Q}_\ell)$ where the action of the subgroup $H(\mathbb{Q}_\ell)$ is now twisted by $\chi^{-1} \circ \nu$.

Using the theory of Shalika models, we will prove the following theorem:

Theorem 4.1. *Let $\pi = \mathbf{1} \boxtimes \sigma$ be as above. Then there exists a locally constant compactly-supported function $\phi\colon \widetilde{G}(\mathbb{Q}_\ell) \to \mathbb{Z}$ satisfying the following:*

(1) For any choice of finite-order unramified character $\chi\colon \mathbb{Q}_\ell^\times \to \mathbb{C}^\times$, any element $\mathfrak{z} \in \mathrm{Hom}_H(\pi \boxtimes \chi^{-1}, \mathbb{C})$ and any spherical vector $\varphi \in \pi \boxtimes \chi^{-1}$, we have the relation

$$\mathfrak{z}(\phi \cdot \varphi) = \frac{\ell^{n^2}}{\ell - 1} L(\sigma \otimes \chi, 1/2)^{-1} \mathfrak{z}(\varphi)$$

where $L(\sigma \otimes \chi, s)$ is the standard local L-factor attached to $\sigma \otimes \chi$.

(2) The function ϕ is an (integral) linear combination of indicator functions $\phi = \sum_{r=0}^{n} b_r \operatorname{ch}((g_r, 1)K)$ where

$$g_r = 1 \times \begin{pmatrix} 1 & \ell^{-1} X_r \\ & 1 \end{pmatrix}$$

and $X_r = \mathrm{diag}(1, \ldots, 1, 0, \ldots, 0)$ is the diagonal matrix having 1 as its first r-entries (we set $X_0 = 0$) and $K = \widetilde{G}(\mathbb{Z}_\ell)$.

(3) If we set $K_1 = G(\mathbb{Z}_\ell) \times (1 + \ell\mathbb{Z}_\ell)$ and

$$V_{1,r} := (g_r, 1) \cdot K_1 \cdot (g_r, 1)^{-1} \cap H(\mathbb{Q}_\ell)$$

then $V_{1,r}$ is contained in $H(\mathbb{Z}_\ell)$ and the coefficients in (2) satisfy

$$(\ell - 1) \cdot b_r \cdot [H(\mathbb{Z}_\ell) : V_{1,r}]^{-1} \in \mathbb{Z}.$$

4.1. **Shalika models.** Let Q denote the Siegel parabolic subgroup of G given by

$$Q = \mathrm{GL}_1 \times \begin{pmatrix} * & * \\ & * \end{pmatrix}$$

and let Q_0 denote its projection to G_0. We denote the unipotent radical of Q (resp. Q_0) by N (resp. N_0) and define the Shalika subgroup $S_0 \subset G_0$ to be the subgroup of all matrices of the form

$$\begin{pmatrix} h & hX \\ & h \end{pmatrix}$$

where $h \in \mathrm{GL}_n$ and $X \in M_n$ (the space of n-by-n matrices). Let $\psi_0 : \mathbb{Q}_\ell \to \mathbb{C}^\times$ denote the standard additive character on $\mathbb{Q}_\ell$ which is trivial on $\mathbb{Z}_\ell$ and define ψ to be the character $S_0(\mathbb{Q}_\ell) \to \mathbb{C}^\times$ given by

$$\psi \begin{pmatrix} h & hX \\ & h \end{pmatrix} = \psi_0(\operatorname{tr} X).$$

Definition 4.2. We say that σ has a *Shalika model* if there exists a non-zero $G_0(\mathbb{Q}_\ell)$-equivariant homomorphism

$$\mathcal{S}_\bullet \colon \sigma \to \mathrm{Ind}_{S_0(\mathbb{Q}_\ell)}^{G_0(\mathbb{Q}_\ell)}(\psi).$$

Shalika models are unique when they exist (see [JR96]) and their existence is closely related to whether σ is $H_0(\mathbb{Q}_\ell)$-distinguished or not. Indeed, we have the following:

Proposition 4.3. *If $\mathrm{Hom}_{H_0}(\sigma, \mathbb{C}) \neq 0$, then σ admits a Shalika model.*

Proof. Let B_0 denote the standard Borel subgroup of G_0. Let w_n denote the longest Weyl element in $\mathrm{GL}_n(\mathbb{Q}_\ell)$, i.e. the anti-diagonal matrix with all entries equal to 1, and consider the $2n \times 2n$ block matrix

$$u = \begin{pmatrix} 1 & \\ w_n & 1 \end{pmatrix}.$$

A routine calculation shows that the $B_0(\mathbb{Q}_\ell)$-orbit Y of $[u] \in H_0(\mathbb{Q}_\ell)\backslash G_0(\mathbb{Q}_\ell) =: X$ is open, dense and the stabiliser $B_u = B_0(\mathbb{Q}_\ell) \cap u^{-1}H_0(\mathbb{Q}_\ell)u$ of $[u]$ is equal to the subgroup

$$B_u = \left\{ \begin{pmatrix} t & \\ & w_n t w_n \end{pmatrix} : t = \mathrm{diag}(t_1, \ldots, t_n) \text{ and } t_i \in \mathbb{Q}_\ell^\times \right\}.$$

Suppose first that σ is a principal series representation, i.e. $\sigma = \mathrm{Ind}_{B_0(\mathbb{Q}_\ell)}^{G_0(\mathbb{Q}_\ell)} \mu\delta_{B_0}^{1/2}$ where μ is an unramified character of the standard torus in $G_0(\mathbb{Q}_\ell)$. Then, by applying Frobenius reciprocity twice we have

$$\mathrm{Hom}_{H_0}(\sigma, \mathbb{C}) = \mathrm{Hom}_{G_0}(\sigma, \mathrm{C}^\infty(X)) = \mathrm{Hom}_{G_0}(\mathrm{C}_c^\infty(X), \sigma^\vee) = \mathrm{Hom}_{B_0}(\mathrm{C}_c^\infty(X), \mu^{-1}\delta_{B_0}^{1/2}).$$

For the last equality we have used the fact that $\sigma^\vee \cong \mathrm{Ind}_{B_0(\mathbb{Q}_\ell)}^{G_0(\mathbb{Q}_\ell)} \mu^{-1}\delta_{B_0}^{1/2}$. If this space is non-zero then we must have $\mathrm{Hom}_{B_0}(C_c^\infty(Y), \mu^{-1}\delta_{B_0}^{1/2}) \neq 0$, which implies that

$$\mu|_{B_u} = \mu\delta_{B_0}^{1/2}|_{B_u} = \delta_{B_u} = 1$$

by [Sak08, §4.4.1] (one checks that $\delta_{B_0}(B_u) = 1$). This implies that there exists a Weyl element w such that ${}^w\mu^{-1}(t) = 1$ for all $t = \mathrm{diag}(t_1, \ldots, t_n, t_1, \ldots, t_n)$ and thus σ has a Shalika model by applying Proposition 1.3 in [AG94].

In general, let σ be the unique unramified quotient of $\sigma' = \mathrm{Ind}_{B_0(\mathbb{Q}_\ell)}^{G_0(\mathbb{Q}_\ell)} \mu\delta_{B_0}^{1/2}$, where μ is unramified. Let $K = \mathrm{GL}_{2n}(\mathbb{Z}_\ell)$ denote the standard maximal compact open subgroup of $G_0(\mathbb{Q}_\ell)$. Since ψ is trivial on $S_0(\mathbb{Q}_\ell) \cap K$, it is enough to show that

$$\mathrm{Hom}_{H_0}(\sigma', \mathbb{C}) \neq 0 \Rightarrow \mathrm{Hom}_{S_0}(\sigma', \psi) \neq 0.$$

But this follows from the above argument, using the fact that one still has an isomorphism $(\mathrm{Ind}_{B_0(\mathbb{Q}_\ell)}^{G_0(\mathbb{Q}_\ell)} \mu\delta_{B_0}^{1/2})^\vee \cong \mathrm{Ind}_{B_0(\mathbb{Q}_\ell)}^{G_0(\mathbb{Q}_\ell)} \mu^{-1}\delta_{B_0}^{1/2}$ even if the representation is not irreducible. □

From now on, we assume that σ admits a Shalika model. This will allow us to define certain *zeta integrals* that compute the local L-factors of σ (and of all its twists by unramified characters). Let dx denote the fixed Haar measure on $\mathrm{GL}_n(\mathbb{Q}_\ell)$ normalised so that the volume of $\mathrm{GL}_n(\mathbb{Z}_\ell)$ is equal to 1.

Proposition 4.4. *Let $\chi\colon \mathbb{Q}_\ell^\times \to \mathbb{C}^\times$ be any continuous character. For $\mathrm{Re}(s)$ large enough, the integral*

$$Z(\varphi, \chi; s) := \int_{\mathrm{GL}_n(\mathbb{Q}_\ell)} \mathcal{S}_\varphi \begin{pmatrix} x & \\ & 1 \end{pmatrix} \chi(\det x)|\det x|^{s-1/2} dx$$

converges. The lower bound on $\mathrm{Re}(s)$ can be chosen independently of φ. Furthermore, the function $Z(\varphi, \chi; s)$ can be analytically continued to a holomorphic function on all of $\mathbb{C}$, and if φ_0 is a spherical vector such that $\mathcal{S}_{\varphi_0}(1) = 1$ then

$$Z(\varphi_0, \chi; s) = L(\sigma \otimes \chi, s).$$

Proof. See [DJR18, §3.2]. □

The zeta integral above provides us with a linear model for σ.

Proposition 4.5. *Let $\chi\colon \mathbb{Q}_\ell^\times \to \mathbb{C}^\times$ be a finite-order unramified character. Then the linear map given by $\varphi \mapsto Z(\varphi, \chi; 1/2)$ constitutes a basis of*

$$\mathrm{Hom}_H(\pi \boxtimes \chi^{-1}, \mathbb{C}). \tag{4.6}$$

Proof. Since the underlying spaces for σ, π and $\pi \boxtimes \chi^{-1}$ are the same, the linear map in the statement of the proposition is well-defined. It is straightforward to verify that

$$Z(h \cdot \varphi, \chi; s) = \chi(\nu(h))\, |\nu(h)|^{s-1/2}\, Z(\varphi, \chi; s)$$

for any $h = (h_1, h_2) \in H(\mathbb{Q}_\ell)$. Taking $s = 1/2$ gives the desired H-equivariance, and the map is clearly non-zero. Finally, note that $\mathrm{Hom}_H(\pi \boxtimes \chi^{-1}, \mathbb{C}) = \mathrm{Hom}_{H_0}(\sigma, \chi \circ \nu)$ and $\chi \circ \nu$ is the product of two characters γ_0, γ_1, one for each factor of $H_0(\mathbb{Q}_\ell)$, each given by the composition of det with χ^{-1}, χ respectively. The result now follows from [CS20]; since χ has finite-order, the character $\gamma_0 \boxtimes \gamma_1$ is always "good" and therefore the Hom space is at most one-dimensional (c.f. the terminology preceding Theorem C and the discussion that follows in *op.cit.*). □

4.2. **Recap on lattices and flag varieties.** In this subsection we introduce the objects that will be used throughout the rest of the section. Let M denote the submonoid of all matrices in $\mathrm{GL}_n(\mathbb{Q}_\ell)$ with entries in $\mathbb{Z}_\ell$, and let $M^{\geq r}$ be the subset of all matrices in M whose determinant has ℓ-adic valuation $\geq r$. For brevity, we set $K_n = \mathrm{GL}_n(\mathbb{Z}_\ell)$.

Definition 4.7. Let $\mathcal{L}_n$ denote the set of all lattices inside $\mathbb{Q}_\ell^n$, i.e. all $\mathbb{Z}_\ell$-submodules which are free of rank n. The set $\mathcal{L}_n$ can be identified with the quotient space $K_n \backslash \mathrm{GL}_n(\mathbb{Q}_\ell)$ via the map

$$\begin{aligned} K_n \backslash \mathrm{GL}_n(\mathbb{Q}_\ell) &\xrightarrow{\sim} \mathcal{L}_n \\ K_n \cdot g &\mapsto L[g]. \end{aligned}$$

where $L[g]$ is the lattice spanned by the rows of g (we view elements of $\mathbb{Q}_\ell^n$ as row vectors). For a lattice $L \in \mathcal{L}_n$, let

$$M_L = \{g \in \mathrm{GL}_n(\mathbb{Q}_\ell) : L[g] \subset L\}$$

denote the subset of all matrices in $\mathrm{GL}_n(\mathbb{Q}_\ell)$ whose rows span a sublattice inside L. This contains the same amount of information as the original lattice L, but it will be useful to consider the subset M_L later on. In particular, $M_{L[g]}$ is simply equal to the translate Mg.

Let $\mathcal{L}_n^\circ \subset \mathcal{L}_n$ denote the finite subset of all lattices satisfying

$$\ell\mathbb{Z}_\ell^n \subset L \subsetneq \mathbb{Z}_\ell^n$$

which is closed under intersection. This can be of course identified with all proper subspaces of $\mathbb{F}_\ell^n$, a perspective we will frequently take.

Definition 4.8. A chain C of lattices in $\mathcal{L}_n$ of length r is a sequence

$$L_r \subsetneq L_{r-1} \subsetneq \cdots \subsetneq L_0.$$

We let $C_{\min} = L_r$ denote the minimal element of the chain C.

Let B_n and T_n denote the standard Borel and torus in GL_n respectively, and let $X_*(T_n)$ denote the cocharacter group $\mathrm{Hom}(\mathbb{G}_m, T_n)$. Fixing this choice of Borel and torus determines a subset $X_*(T_n)^+ \subset X_*(T_n)$ of dominant cocharacters. Explicitly, every dominant cocharacter is of the form

$$\begin{aligned} \mathbb{G}_m &\longrightarrow T_n \\ z &\mapsto \mathrm{diag}(z^{a_1}, \ldots, z^{a_n}) \end{aligned}$$

where a_i are integers satisfying $a_1 \geq \cdots \geq a_n$, therefore we will often refer to such a cocharacter by the tuple $(a_1, \ldots, a_n)$. Recall that one has the Cartan decomposition of $\mathrm{GL}_n(\mathbb{Q}_\ell)$

$$\mathrm{GL}_n(\mathbb{Q}_\ell) = \bigsqcup_\chi K_n \chi(\ell) K_n$$

where the disjoint union is over all dominant cocharacters as above.

Definition 4.9. For a lattice $L = L[g]$ in $\mathcal{L}_n$, we let $\mathrm{inv}(L)$ denote its relative position with respect to the standard lattice $\mathbb{Z}_\ell^n$, i.e. $\mathrm{inv}(L) = \chi(\ell)$ where χ is the unique dominant cocharacter as above, such that $K_n g K_n = K_n \chi(\ell) K_n$.

We introduce the following matrices which will play a key role in the rest of this section.

Definition 4.10. For $1 \leq m \leq n$, let

$$t_m = \mathrm{diag}(\ell, \ldots, \ell, 1, \ldots, 1) \in \mathrm{GL}_n(\mathbb{Q}_\ell)$$

denote the diagonal matrix whose first m entries are equal to ℓ.

Remark 4.11. The space $\mathcal{L}_n^\circ$ can be viewed as the subset of $\mathcal{L}_n$ generated by all intersections of lattices of relative position t_1, or equivalently, as the subset of all lattices with relative position equal to t_m, for some $1 \leq m \leq n$.

We will also need to talk about flag varieties of arbitrary signature, so we recall the definition for the convenience of the reader.

Definition 4.12. Let $d_0 < d_1 < \cdots < d_k$ be positive integers satisfying $d_0 = 0$ and $d_k = m$. The flag variety $\mathrm{Fl}(d_1, \dots, d_k; \mathbb{F}_\ell)$ of signature $(d_1, \dots, d_k)$ is the space of all flags

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_k = \mathbb{F}_\ell^m$$

where V_i is a subspace of dimension d_i.

We will frequently use the fact that $\mathrm{Fl}(d_1, \dots, d_k; \mathbb{F}_\ell)$ is a finite set of order equal to

$$\begin{bmatrix} m \\ n_1 \cdots n_k \end{bmatrix}_\ell := \frac{[m]_\ell!}{[n_1]_\ell! \cdots [n_k]_\ell!}$$

where $n_i = d_i - d_{i-1}$, with notation as in section 1.2.

4.3. **Lattice counting.** To study the zeta integrals introduced in Proposition 4.4 and their translates by certain Hecke operators, it will be necessary to formulate an inclusion-exclusion principle for the monoid M. Essentially, we will use this principle to count the elements in $M^{\geq 1}$ with respect to a suitable measure arising from the Shalika model for π_0.

Fix a left-invariant Haar measure on $\mathrm{GL}_n(\mathbb{Q}_\ell)$ (which must also be right-invariant), and let $f\colon \mathrm{GL}_n(\mathbb{Q}_\ell) \to \mathbb{C}$ be any locally constant function such that $\int_{\mathrm{GL}_n(\mathbb{Q}_\ell)} f(x)dx < +\infty$. Then the assignment

$$X \mapsto \mu_f(X) := \int_X f(x)dx$$

defines a (finite) measure on the σ-algebra generated by all compact open subsets of $\mathrm{GL}_n(\mathbb{Q}_\ell)$.

Proposition 4.13 (Inclusion-Exclusion). *Let $f\colon \mathrm{GL}_n(\mathbb{Q}_\ell) \to \mathbb{C}$ be any locally constant function. Then*

$$\mu_f\left(M^{\geq 1}\right) = \mu_f\left(\bigcup_{L \in \mathcal{L}_n^\circ} M_L\right) = \sum_{C \in \mathfrak{C}} (-1)^{l(C)} \mu_f\left(M_{C_{\min}}\right) \tag{4.14}$$

where $\mathfrak{C}$ denotes the set of chains in $\mathcal{L}_n^\circ$ and $l(C)$ is the length of the chain C (Definition 4.8).

Proof. We follow [Nar74]. For any non-empty finite subset S of $\mathcal{L}_n^\circ$, we define a subset $\overline{S} \subset \mathcal{L}_n^\circ$ as follows: set $S_0 = S$, and for $i \geq 0$, set $S_{i+1} = S_i \cup \{L \cap L' | L, L' \in S_i\}$. Then we take $\overline{S} = \bigcup_i S_i$. The collection of all such subsets $\overline{S}$ obtained from any finite subset S is denoted by $\mathcal{O}(\mathcal{L}_n^\circ)$. Recall that for any $L \in \mathcal{L}_n^\circ$, we have set

$$M_L = \{g \in \mathrm{GL}_n(\mathbb{Q}_\ell) \,|\, L[g] \subset L\} \subset M^{\geq 1}.$$

Then, $M^{\geq 1} = \bigcup_{L \in \mathcal{L}_n^\circ} M_L$. Indeed, for $g \in M^{\geq 1}$ one has $L[g] \subsetneq \mathbb{Z}_\ell^n$, which implies that $L[g] + \ell\mathbb{Z}_\ell^n \subsetneq \mathbb{Z}_\ell^n$. Therefore, $g \in M_L$ with $L = L[g] + \ell\mathbb{Z}_\ell^n$.

By the inclusion-exclusion principle for finite measures, we have

$$\mu_f\left(M^{\geq 1}\right) = \sum_{\emptyset \neq S \subset \mathcal{L}_n^\circ} (-1)^{|S|-1} \mu_f\left(\bigcap_{L \in S} M_L\right). \tag{4.15}$$

Fix an element $Y \in \mathcal{O}(\mathcal{L}_n^\circ)$ and let

$$Y_0 := Y - \{L \cap L' \,|\, L, L' \in Y, L \not\subset L', L' \not\subset L\}$$

i.e. the subset obtained by deleting the intersection of incomparable pairs of lattices. Set $s = |Y_0|$, $t = |Y|$, $l = t - s$. Clearly, for any non-empty $S \subset \mathcal{L}_n^\circ$, we have $\overline{S} = Y$ if and only if $Y_0 \subset S \subset Y$. Therefore

$$\begin{aligned} \sum_{S \subset \mathcal{L}_n^\circ, \overline{S} = Y} (-1)^{|S|} &= (-1)^s \sum_{k=0}^{l} \binom{l}{k} (-1)^k \\ &= \begin{cases} (-1)^s & \text{if } Y = Y_0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Moreover, $Y = Y_0$ if and only if elements of Y form a chain $C \in \mathfrak{C}$. Using the fact that $\bigcap_{L\in C} M_L = M_{C_{\min}}$, the right hand side of (4.15) is equal to

$$\sum_{C\in\mathfrak{C}} \left(\sum_{S:\,\overline{S}=C} (-1)^{|S|-1} \right) \mu_f(M_{C_{\min}}) = \sum_{C\in\mathfrak{C}} (-1)^{l(C)} \mu_f(M_{C_{\min}})$$

and the claim follows. □

We obtain the following corollary.

Corollary 4.16. *Keeping the same notation as in Proposition 4.13, suppose that f is invariant under right-translation of the argument by K_n. For an integer $1 \le m \le n$ let*

$$\lambda_m = \#\{L \in \mathcal{L}_n^\circ : \mathrm{inv}(L) = t_m\} \cdot \sum_{j=0}^{m-1} (-1)^j \cdot \# \left\{ \begin{array}{c} \text{chains in } \mathcal{L}_n^\circ \text{ of length } j \\ \text{with minimal element } L[t_m] \end{array} \right\}.$$

Then

1. $\mu_f(M^{\ge 1}) = \sum_{m=1}^n \lambda_m \mu_f(Mt_m)$.
2. *We have*

$$\lambda_m \equiv \binom{n}{m} (-1)^{m+1}$$

modulo $\ell - 1$, which implies that $\sum_{m=1}^n \lambda_m \equiv 1$ modulo $\ell - 1$.

Proof. Since f is invariant under right-translation by K_n, by Proposition 4.13 and the fact that any element in $\mathcal{L}_n^\circ$ has relative position t_m for some m (Remark 4.11), we have

$$\begin{aligned} \mu_f(M^{\ge 1}) &= \sum_{m=1}^n \sum_{\mathrm{inv}(L)=t_m} \sum_{j=0}^{m-1} (-1)^j D_{j,m} \cdot \mu_f(M_L) \\ &= \sum_{m=1}^n \lambda_m \cdot \mu_f(Mt_m) \end{aligned}$$

where $D_{j,m}$ is the number of chains of length j with maximal element $L[t_m]$. To calculate $D_{j,m}$ it is enough to study subspaces of $\mathbb{F}_\ell^m$ (since $\mathcal{L}_n^\circ$ can be identified with all proper subspaces of $\mathbb{F}_\ell^n$).

Note that $D_{j,m}$ is equal to the sum over $(d_1, \ldots, d_{j+1})$ of $\# \mathrm{Fl}(d_1, \ldots, d_{j+1}; \mathbb{F}_\ell)$, where the latter space is the flag variety of signature $(d_1, \ldots, d_{j+1})$ associated to the vector space $\mathbb{F}_\ell^m$ (Definition 4.12). The order of this flag variety is congruent modulo $\ell - 1$ to the multinomial coefficient

$$\binom{m}{n_1 \ \cdots \ n_{j+1}}$$

where $n_i = d_i - d_{i-1}$ (and $d_0 = 0$). This implies that $D_{j,m}$ is congruent to the sum of these coefficients over all n_i such that $n_i \neq 0$ for all $i = 1, \ldots, j+1$. The total sum of all coefficients is $(j+1)^m$ by the multinomial theorem. We compute $D_{j,m}$ as follows:

$$D_{j,m} \equiv (j+1)^m - \sum_{\text{some } n_i=0} = (j+1)^m - \left(\sum_i \sum_{n_i=0} - \sum_{i,j} \sum_{n_i=n_j=0} + \cdots \pm \sum_{\text{all } n_i=0} \right).$$

This is equal to

$$D_{j,m} \equiv (j+1)^m - \left(\binom{j+1}{1} j^m - \binom{j+1}{2} (j-1)^m + \cdots \pm 0 \right) = \sum_{k=0}^{j} (-1)^k \binom{j+1}{k} (j+1-k)^m.$$

Consider the following variable change $(r,t) = (j-k,k)$. This implies that

$$\begin{aligned}\sum_{j=0}^{m-1}(-1)^j D_{j,m} &\equiv \sum_{r=0}^{m-1}(-1)^r(r+1)^m \sum_{t=0}^{m-r-1}\binom{r+t+1}{t} \\ &= \sum_{r=0}^{m-1}(-1)^r(r+1)^m\binom{m+1}{m-r-1} \\ &= -\sum_{s=0}^{m-1}(-1)^s\binom{m+1}{s}P(s)\end{aligned}$$

where $P(X) = (X-m)^m$ and for the last equality we have used the change of variables $s = m-r-1$. Now we have $P(m) = 0$ and $P(m+1) = 1$, and since P is a polynomial of degree $< m+1$, by the standard identities involving binomial coefficients we have

$$-\sum_{s=0}^{m-1}(-1)^s\binom{m+1}{s}P(s) = (-1)^{m+1}P(m+1) = (-1)^{m+1}.$$

To complete the proof we note that $\#\{L \in \mathcal{L}_n^\circ : \mathrm{inv}(L) = t_m\}$ is equal to the number of points in the Grassmanian of $(n-m)$-dimensional subspaces of $\mathbb{F}_\ell^n$. This quantity is congruent to

$$\binom{n}{n-m} = \binom{n}{m}$$

modulo $\ell - 1$. □

Example 4.1. Let $n = 2$ and suppose that f is right K_n-invariant. Then there are $\ell+2$ elements in $\mathcal{L}_2^\circ$; $\ell+1$ of them have relative position t_1 and we will denote them by $L_1, \ldots, L_{\ell+1}$, the other has relative position t_2, and we will denote it by S. There are $\ell+2$ chains of length zero and $\ell+1$ chains of length 1, namely they are all of the form $S \subset L_i$. Since f is K_n-invariant we have

$$\mu_f(M^{\geq 1}) = (\ell+1)\mu_f(Mt_1) + \mu_f(Mt_2) - (\ell+1)\mu_f(Mt_2) = (\ell+1)\mu_f(Mt_1) - \ell\mu_f(Mt_2).$$

4.4. **The U_m-operators.** Let $\mathrm{M}_{m\times n}(\mathbb{F}_\ell)$ denote the set of $(m\times n)$-matrices with coefficients in $\mathbb{F}_\ell$ which can naturally be identified with the subgroup of $\mathrm{M}_{n\times n}(\mathbb{F}_\ell)$ consisting of matrices with the last $n-m$ rows equal to zero. We also view $\mathrm{M}_{n\times n}(\mathbb{F}_\ell) \subset \mathrm{M}_{n\times n}(\mathbb{Z}_\ell)$ via the Teichmüller lift. We emphasise that when talking about the ranks of these matrices, we always mean the rank as a linear map $\mathbb{F}_\ell^n \to \mathbb{F}_\ell^m$ (as the Teichmüller lift doesn't preserve ranks).

We introduce some operators. For $1 \leq m \leq n$ we let

$$U_m = \sum_X \begin{pmatrix} 1 & X \\ & 1 \end{pmatrix}\begin{pmatrix} t_m & \\ & 1 \end{pmatrix} \in \mathbb{Z}[G_0(\mathbb{Q}_\ell)]$$

where the sum is over all matrices in $\mathrm{M}_{m\times n}(\mathbb{F}_\ell)$ (the blocks in the above matrices are of size $n \times n$). These operators interact with the Shalika model as follows.

Lemma 4.17. *Let φ_0 be a spherical vector. Then*

$$\mathcal{S}_{U_m\cdot\varphi_0}\begin{pmatrix} x & \\ & 1 \end{pmatrix} = \begin{cases} 0 & \text{if } x \notin M \\ \ell^{mn}\mathcal{S}_{\varphi_0}\begin{pmatrix} xt_m & \\ & 1 \end{pmatrix} & \text{if } x \in M \end{cases}$$

Consequently, we have

$$Z(\chi(\ell)^m \ell^{n(n-m)} U_m \cdot \varphi_0, \chi; s) = \ell^{n^2+m(s-1/2)} \int_{Mt_m} S_{\varphi_0}\begin{pmatrix} x & \\ & 1 \end{pmatrix} \chi(\det x)|\det x|^{s-1/2}dx$$

for $\mathrm{Re}(s)$ *large enough so that the integrals converge.*

Proof. Right-translation by elements of the form $\begin{pmatrix} 1 & X \\ & 1 \end{pmatrix}$ implies that $\mathcal{S}_{\varphi_0}\begin{pmatrix} x & \\ & 1 \end{pmatrix}$ is supported on M. Furthermore, a direct calculation shows that

$$\mathcal{S}_{U_m \cdot \varphi_0}\begin{pmatrix} x & \\ & 1 \end{pmatrix} = \sum_X \psi_0(\mathrm{tr}(x \cdot X)) \cdot \mathcal{S}_{\varphi_0}\begin{pmatrix} xt_m & \\ & 1 \end{pmatrix}$$

so this function is supported on all $x \in Mt_m^{-1}$. If $x \in Mt_m^{-1}$, then the function $X \mapsto \psi_0(\mathrm{tr}(x \cdot X))$ defines a character $\mathrm{M}_{m \times n}(\mathbb{F}_\ell) \to \mathbb{C}$ which is trivial if and only if $x \in M$. Therefore, the result follows from the usual character orthogonality relations. The last part follows from the change of variable $x \mapsto xt_m^{-1}$. □

Remark 4.18. The action of U_m on the function $\mathcal{S}_{\varphi_0}(-)$ differs from the action of the more common U_ℓ-operator associated with the parabolic of type $(m, n-m, n)$ by a power of ℓ. However, it is more convenient for our purposes to use the above definition instead. Since we are eventually going to consider congruences modulo $\ell - 1$ this makes no difference to the end result.

We now apply the combinatorics of the previous section.

Theorem 4.19. *Let $\chi : \mathbb{Q}_\ell^\times \to \mathbb{C}^\times$ be a finite-order unramified character and let φ_0 be a spherical vector normalised such that $\mathcal{S}_{\varphi_0}(1) = 1$. Then there exist integers $\lambda_1, \ldots, \lambda_n$ (independent of χ) such that*

$$Z\left((\sum_{m=1}^{n} \lambda_m \ell^{n(n-m)} \chi(\ell)^m U_m) \cdot \varphi_0, \chi; 1/2\right) = \ell^{n^2}\left[Z(\varphi_0, \chi; 1/2) - 1\right].$$

Furthermore, the sum $\ell^{n^2} - \sum_{m=1}^{n} \ell^{n(n-m)}\lambda_m$ is divisible by $\ell - 1$.

Proof. Fix s with real part large enough, and set $f(x) = \mathcal{S}_{\varphi_0}\begin{pmatrix} x & \\ & 1 \end{pmatrix}\chi(\det x)|\det x|^{s-1/2}$. Then f is right K_n-invariant. By Corollary 4.16, there exist integers $\lambda_1, \ldots, \lambda_n$ which are independent of χ and satisfy

$$\mu_f(M^{\geq 1}) = \sum_{m=1}^{n} \lambda_m \mu_f(Mt_m).$$

Therefore, by Lemma 4.17, we have

$$\begin{aligned} Z\left((\sum_{m=1}^{n} \lambda_m \ell^{n(n-m)} \chi(\ell)^m U_m) \cdot \varphi_0, \chi; s\right) &= \sum_{m=1}^{n} \lambda_m \ell^{n^2 + m(s-1/2)} \mu_f(Mt_m) \\ &= \ell^{n^2}\left(\sum_{m=1}^{n} \lambda_m (\ell^{m(s-1/2)} - 1)\mu_f(Mt_m) + \mu_f(M^{\geq 1})\right). \end{aligned}$$

Furthermore, $\mu_f(Mt_m)$ can be analytically continued to a holomorphic function in s, so the limit $\lim_{s \to 1/2} \mu_f(Mt_m)$ exists and is finite. Therefore passing to the limit as $s \to 1/2$ we have

$$Z\left((\sum_{m=1}^{n} \lambda_m \ell^{n(n-m)} \chi(\ell)^m U_m) \cdot \varphi_0, \chi; 1/2\right) = \ell^{n^2} \lim_{s \to 1/2} \mu_f(M^{\geq 1}).$$

Now to conclude the proof, we note that

$$\mu_f(M^{\geq 1}) = \mu_f(M) - \mu_f(K_n) = Z(\varphi_0, \chi; s) - 1.$$

The second part of the theorem just follows from Corollary 4.16. □

4.5. **Congruences modulo $\ell - 1$.** We now seek to find the element ϕ in Theorem 4.1. For this, we must prove a congruence relation for the operators U_m. As before, we will express the result in terms of the extended group $\widetilde{G}$, so we introduce some additional notation:

Definition 4.20. For an element $g \in G_0(\mathbb{Q}_\ell)$ let $\widetilde{g}$ denote the element

$$(1, g, \det g^{-1}) \in \widetilde{G}(\mathbb{Q}_\ell).$$

We also let $\widetilde{U}_m$ denote the corresponding operator in $\mathbb{Z}[\widetilde{G}(\mathbb{Q}_\ell)]$.

We note that we have a left action of $H(\mathbb{Z}_\ell) = \mathrm{GL}_1(\mathbb{Z}_\ell) \times \mathrm{GL}_n(\mathbb{Z}_\ell) \times \mathrm{GL}_n(\mathbb{Z}_\ell)$ on $\mathrm{M}_{n\times n}(\mathbb{F}_\ell)$ given by

$$(\alpha, A, B) \cdot X = AXB^{-1}$$

which will be useful in the proof of the following proposition.

Proposition 4.21. *We retain the notation of Theorem 4.1. Let $\varphi_0 \in \pi$ be a spherical vector. Then there exists a $\mathbb{Z}$-valued right K-invariant smooth compactly supported function ψ_m on $\widetilde{G}(\mathbb{Q}_\ell)$ such that*

(1) $\psi_m = \sum_{r=0}^m c_{r,m} \operatorname{ch}((g_r, 1)K)$ and the coefficients $c_{r,m}$ equal the number of rank r matrices in $\mathrm{M}_{m\times n}(\mathbb{F}_\ell)$; explicitly we have

$$c_{r,m} = \prod_{j=0}^{r-1} \frac{(\ell^m - \ell^j)(\ell^n - \ell^j)}{\ell^r - \ell^j}$$

for $r \geq 1$, and $c_{0,m} = 1$.

(2) For any unramified χ and $\mathfrak{z} \in \mathrm{Hom}_H(\pi \boxtimes \chi^{-1}, \mathbb{C})$

$$\mathfrak{z}(\widetilde{U}_m \cdot \varphi_0) = \mathfrak{z}(\psi_m \cdot \varphi_0).$$

(3) For $r \geq 0$ we have

$$\mathrm{Stab}_{H(\mathbb{Z}_\ell)}(X_r) = V_r := (g_r, 1)K(g_r, 1)^{-1} \cap H(\mathbb{Q}_\ell)$$

which implies that $[H(\mathbb{Z}_\ell) : V_r] = c_{r,n}$. Furthermore, we have $[V_r : V_{1,r}] = \ell - 1$ when $0 \leq r \leq n-1$ and $V_n = V_{1,n}$.

Proof. Let $g_X = \begin{pmatrix} 1 & X \\ & 1 \end{pmatrix}\begin{pmatrix} t_m & \\ & 1 \end{pmatrix}$ be an element appearing the sum U_m. Since φ_0 is spherical, we have

$$g_X \cdot \varphi_0 = \begin{pmatrix} t_m A^{-1} & \\ & B^{-1} \end{pmatrix}\begin{pmatrix} 1 & t_m^{-1}(t_m A t_m^{-1}) X B^{-1} \\ & 1 \end{pmatrix} \cdot \varphi_0$$

for $A, B \in \mathrm{GL}_n(\mathbb{Z}_\ell)$. We can choose A, B such that $t_m A t_m^{-1} \in \mathrm{GL}_n(\mathbb{Z}_\ell)$ and $(t_m A t_m^{-1}) X B^{-1} \equiv X_r$ modulo ℓ, where r is the rank of X (viewed as a linear map $\mathbb{F}_\ell^n \to \mathbb{F}_\ell^m$).[3] Hence we have

$$\mathfrak{z}(\widetilde{g}_X \cdot \varphi_0) = \mathfrak{z}((g_r, 1) \cdot \varphi_0).$$

Indeed, this follows from that fact that $(1, (t_m A^{-1}, B^{-1}), \ell^{-m} \det A \det B^{-1}) \in H(\mathbb{Q}_\ell)$ and $\chi(\det A) = \chi(\det B) = 1$. This proves the first and second parts of the proposition – the formula for $c_{r,m}$ is well-known.

The last part of the proposition follows from an explicit calculation and the orbit–stabiliser theorem. Furthermore the induced map

$$\nu \colon V_r \to \mathbb{Z}_\ell^\times$$

given by $\nu(\alpha, A, B) = \det A^{-1} \det B$ is surjective when $r < n$ (for example, take $A = 1$ and B equal to a diagonal matrix with the first $n-1$ entries equal to 1) so we have $[V_r : V_{1,r}] = \ell - 1$. □

4.6. **Proof of Theorem 4.1.** We begin with the following lemma:

Lemma 4.22. *For $1 \leq r \leq n$ set*

$$b'_r = \sum_{m=r}^{n} \ell^{n(n-m)} \lambda_m c_{r,m}$$

where λ_m and $c_{r,m}$ are the integers in Corollary 4.16 and Proposition 4.21 respectively. Then $(\ell - 1) \cdot c_{r,n}$ divides b'_r when $1 \leq r \leq n-1$, and $b'_n = \lambda_n c_{n,n}$.

[3] Working modulo ℓ, we can first apply column operations to ensure that X has its last $(n-r)$ columns equal to zero. Then apply row operations to ensure that X is only non-zero in its top-left $(r \times r)$ block (such an A exists for this step because $r \leq m$). This block must be invertible, so we can apply column operations to put X in the form X_r, as required.

Proof. We first show that $c_{r,n}$ divides $\lambda_m c_{r,m}$. Indeed the number of $\mathbb{F}_\ell$-points of the Grassmanian of m-dimensional subspaces of n-dimensional space is equal to $\begin{bmatrix} n \\ m \end{bmatrix}_\ell$, so by Corollary 4.16 the integer λ_m is divisible by $\begin{bmatrix} n \\ m \end{bmatrix}_\ell$. Using the formula in Proposition 4.21 we have

$$\begin{bmatrix} n \\ m \end{bmatrix}_\ell \cdot \frac{c_{r,m}}{c_{r,n}} = \begin{bmatrix} n \\ m \end{bmatrix}_\ell \cdot \begin{bmatrix} m \\ r \end{bmatrix}_\ell \cdot \begin{bmatrix} n \\ r \end{bmatrix}_\ell^{-1} = \begin{bmatrix} n-r \\ n-m \end{bmatrix}_\ell .$$

Furthermore, by Corollary 4.16 we have $\lambda_m \equiv \binom{n}{m}(-1)^{m+1}$ modulo $\ell - 1$, which implies that

$$\ell^{n(n-m)} \cdot \lambda_m \cdot c_{r,m} \cdot c_{r,n}^{-1} \equiv (-1)^{m+1} \binom{n-r}{n-m} \text{ modulo } \ell - 1$$

and thus by the binomial formula, $b'_r c_{r,n}^{-1} \equiv 0 \pmod{\ell - 1}$ in the case $r < n$. This completes the proof of the lemma. □

We now construct the element ϕ. We set

$$b_0 = (\ell-1)^{-1}\left(\ell^{n^2} - \sum_{m=1}^{n} \lambda_m \ell^{n(n-m)}\right)$$

which is an integer by Theorem 4.19, and for $r \geq 1$ we set $b_r = -(\ell-1)^{-1} b'_r$ which is an integer by the above lemma and the fact that $\ell - 1$ divides $c_{n,n}$. Then we set

$$\phi := \sum_{r=0}^{n} b_r \operatorname{ch}((g_r, 1)K) \quad \in \quad \mathcal{H}\left(\widetilde{G}(\mathbb{Q}_\ell), \mathbb{Z}\right)$$

which naturally acts on the smooth representation $\pi \boxtimes \chi^{-1}$ of $\widetilde{G}(\mathbb{Q}_\ell)$. Let φ_0 be a normalised spherical vector and set $\mathfrak{z}(-) = Z(-, \chi; 1/2)$. Then we have

$$\begin{aligned} \mathfrak{z}((\ell-1)\phi \cdot \varphi_0) &= \mathfrak{z}\left(\left(\ell^{n^2} - \sum_{m=1}^{n} \lambda_m \ell^{n(n-m)}\right)\varphi_0 - \sum_{r=1}^{n} b'_r (g_r, 1) \cdot \varphi_0\right) \\ &= \mathfrak{z}\left(\left(\ell^{n^2} - \sum_{m=1}^{n} \lambda_m \ell^{n(n-m)}\right)\varphi_0 - \sum_{r=1}^{n}\sum_{m=r}^{n} \ell^{n(n-m)}\lambda_m c_{r,m}(g_r, 1) \cdot \varphi_0\right) \\ &= \mathfrak{z}\left(\ell^{n^2}\varphi_0 - \sum_{m=1}^{n} \ell^{n(n-m)}\lambda_m \sum_{r=0}^{m} c_{r,m}(g_r, 1)\cdot\varphi_0\right) \\ &= \mathfrak{z}\left(\ell^{n^2}\varphi_0 - \sum_{m=1}^{n} \ell^{n(n-m)}\lambda_m \cdot \widetilde{U}_m \cdot \varphi_0\right) \qquad (4.23) \\ &= Z\left(\ell^{n^2}\varphi_0 - (\sum_{m=1}^{n} \lambda_m \ell^{n(n-m)}\chi(\ell)^m U_m)\cdot\varphi_0, \chi; 1/2\right) \qquad (4.24) \\ &= \ell^{n^2} L(\sigma\otimes\chi, 1/2)^{-1} Z(\varphi_0, \chi; 1/2) \qquad (4.25) \end{aligned}$$

where the equalities in (4.23), (4.24) and (4.25) follow from Proposition 4.21, the fact that $\widetilde{U}_m \cdot \varphi_0 = \chi(\ell)^m U_m \cdot \varphi_0$ and Theorem 4.19 respectively. The latter equality also uses the fact that $Z(\varphi_0, \chi; 1/2) = L(\sigma \otimes \chi, 1/2)$.

Since $Z(-, \chi; 1/2)$ is a basis of the hom-space $\mathrm{Hom}_H(\pi \boxtimes \chi^{-1}, \mathbb{C})$, this completes the proof of the first part of Theorem 4.1. Furthermore, by construction ϕ is an integral linear combination of indicator functions $\operatorname{ch}((g_r, 1)K)$, so we have also proven the second part of the theorem.

Finally, by Proposition 4.21 we have

$$(\ell-1)\cdot b_r \cdot [H(\mathbb{Z}_\ell) : V_{1,r}]^{-1} = \begin{cases} -b'_r \cdot c_{r,n}^{-1} \cdot (\ell-1)^{-1} & \text{if } 1 \leq r \leq n-1 \\ -b'_n \cdot c_{n,n}^{-1} & \text{if } r = n \end{cases}$$

which in both cases was shown to be an integer. For $r = 0$ the statement follows from the fact that b_0 is an integer.

5. Interlude: cohomology functors

In [Loe19], a class of functors on subgroups of locally profinite groups is defined in an abstract setting that captures the key properties of the cohomology of locally symmetric spaces. In this section, we study these functors in more detail, and develop some machinery that will be useful later on.

5.1. Generalities.

Let G be a locally profinite topological group, $\Sigma \subset G$ an open submonoid and let Υ be a collection of compact open subgroups of G contained in Σ. We will impose the following conditions on (G, Σ, Υ)

- (T1) For all $g \in \Sigma \cup \Sigma^{-1}$ and $K \in \Upsilon$, $gKg^{-1} \subset \Sigma$ implies $gKg^{-1} \in \Upsilon$.
- (T2) For all $K \in \Upsilon$, Υ contains a topological basis of open normal subgroups of K.
- (T3) For all $K, L \in \Upsilon$ and $g \in \Sigma \cup \Sigma^{-1}$, we have $K \cap gLg^{-1} \in \Upsilon$.

To such a triplet, we associate a *category of compact opens* $\mathcal{P}(G, \Sigma, \Upsilon)$ whose objects are elements of Υ and whose morphisms are given by

$$\mathrm{Hom}_{\mathcal{P}(G,\Sigma,\Upsilon)}(L, K) := \left\{ g \in \Sigma | \, g^{-1}Lg \subset K \right\}$$

for $L, K \in \Upsilon$, with compositions given by $(L \xrightarrow{g} K) \circ (L' \xrightarrow{h} L) = (L' \xrightarrow{hg} K)$. We will denote a morphism $(L \xrightarrow{g} K)$ by $[g]_{L,K}$; if we have $L \subset K$ and $g = 1$, we shall also denote the corresponding morphism by $\mathrm{pr}_{L,K}$. If no confusion can arise, we will suppress the subscripts from these morphisms. Omission of Υ from the triplet (G, Σ, Υ) will mean that we take all compact open subgroups of Σ, and omission of Σ means $\Sigma = G$.

Definition 5.1. A *cohomology functor* M on a triplet (G, Σ, Υ) as above and valued in R-**Mod** for a commutative ring R is a pair of covariant functors

$$M^\star \colon \mathcal{P}(G, \Sigma, \Upsilon)^{\mathrm{op}} \to R\text{-}\mathbf{Mod} \qquad M_\star \colon \mathcal{P}(G, \Sigma^{-1}, \Upsilon) \to R\text{-}\mathbf{Mod}$$

satisfying the following:

- (C1) $M_\star(K) = M^\star(K)$ for all $K \in \Upsilon$. We denote this common value by $M(K)$.
- (C2) If ϕ is a morphism then we set $\phi_\star = M_\star(\phi)$ and $\phi^\star = M^\star(\phi)$. We require that
$$[g]^\star_{L,K} = [g^{-1}]_{K,L,\star} \in \mathrm{Hom}(M(K), M(L))$$
for all $g \in \Sigma$ and $L, K \in \Upsilon$ satisfying $g^{-1}Lg = K$.
- (C3) $[\gamma]_{K,K,\star} = \mathrm{id}$ for all $\gamma \in K$ and $K \in \Upsilon$.

We will denote the cohomology functor as $M : (G, \Sigma, \Upsilon) \to R$-**Mod**, and sometimes abusively as $M : \mathcal{P}_G \to R$-**Mod** if the categories of compact opens are clear from context.

Our cohomology functors will often be enhanced with the following additional axioms.

Definition 5.2. Let $M : (G, \Sigma, \Upsilon) \to \mathrm{R}$ $-$**Mod** be a cohomology functor.

- (G) We say M is *Galois* if for all $K, L \in \Upsilon$ with $L \triangleleft K$,
$$\mathrm{pr}^\star_{L,K} : M(K) \xrightarrow{\sim} M(L)^{K/L}$$
where the (left) action of $\gamma \in K/L$ on $x \in M(L)$ is via $x \mapsto [\gamma]^\star_{L,L}(x)$.
- (Co) We say that M is a *covering functor* if for all $L, K \in \Upsilon$, $L \subset K$,
$$(L \xrightarrow{\mathrm{pr}} K)_\star \circ (L \xrightarrow{\mathrm{pr}} K)^\star = (K \xrightarrow{[K:L]} K)$$
i.e. the composition is multiplication by index $[K : L]$ on $M(K)$.
- (M) We say M is *Cartesian* if for all $K, L, L' \in \Upsilon$ with $L, L' \subset K$, we have a commutative diagram
$$\begin{array}{ccc} \bigoplus_\gamma M(L_\gamma) & \xrightarrow{\sum \mathrm{pr}_\star} & M(L) \\ {\scriptstyle \sum \gamma^\star}\uparrow & & \uparrow{\scriptstyle \mathrm{pr}^\star} \\ M(L') & \xrightarrow[\mathrm{pr}_\star]{} & M(K) \end{array}$$

where the direct sum in the top left corner is over a fixed choice of coset representatives $\gamma \in L\backslash K/L'$ and $L_\gamma = \gamma L'\gamma^{-1} \cap L \in \Upsilon$. This condition is independent of the representatives, since if γ are replaced with $\gamma' = \delta\gamma\delta'$ for $\delta \in L$, $\delta' \in L'$, L_γ is replaced with $L_{\gamma'} = \delta L_\gamma \delta^{-1}$, and

$$(L_\gamma \xrightarrow{\mathrm{pr}} L)_\star \circ (L_\gamma \xrightarrow{\gamma} L)^\star = (L_{\gamma'} \xrightarrow{\mathrm{pr}} L)_\star \circ (L_{\gamma'} \xrightarrow{\gamma'} L)^\star$$

If M satisfies both (M) and (Co), we will say that M is *CoMack*. It is easily seen that if M is CoMack and R is a $\mathbb{Q}$-algebra, then M is Galois.

Remark 5.3. If one takes Υ to be all compact opens in Σ, then one obtains the definition of a cohomology functor given in [Loe19]. The addition of Υ is made since we only want to vary over sufficiently small compact open subgroups of $\mathbf{G}(\mathbb{A}_f)$ for a reductive group $\mathbf{G}$ (Definition B.5), and the conditions (T1)–(T3) on Υ are included so that taking products is well-behaved. Axiom (G) is inspired by Galois descent in étale cohomology and axiom (Co) reflects the corresponding property covering maps in étale cohomology [AGV73, Tome 3, Expose IX, §5]. Axiom (M) is based on Mackey's *decomposition formula* e.g. see [Dre73]. In [Loe19], this property was named *Cartesian* after the Cartesian condition for proper/smooth base change in étale cohomology. The terminology 'CoMack' is standard e.g. see [TW95].

Example 5.1. Let X be a set with a right action of Σ, and Υ any collection as above. Let $M(K) = C_c^\infty(X/K; R)$ be the set of all functions $\zeta\colon X \to R$ which factor through X/K and are non-zero only on finitely many orbits. For $g \in \Sigma$, we denote by $g \star \zeta$ the composition $X \xrightarrow{g} X \xrightarrow{\zeta} R$, which gives a *left* action on the set of functions from X to R. Then, for elements $\sigma, \tau^{-1} \in \Sigma$ and morphisms $\sigma\colon L \to K$, $\tau\colon L' \to K'$, we set

$$[\sigma]^\star\colon M(K) \to M(L) \qquad \zeta \mapsto \sigma \star \zeta$$
$$[\tau]_\star\colon M(L') \to M(K') \qquad \zeta \mapsto \sum_\gamma \left(\gamma\tau^{-1}\right) \star \zeta$$

where the sum runs over representatives in K' of the coset space $K'/(\tau^{-1}L'\tau)$. Then M is CoMack.

Example 5.2. The prototypical example of a cohomology functor for us will be the cohomology of Shimura varieties as also explained in [Loe19, §3]. Let $(\mathbf{G}, X)$ be a Shimura–Deligne datum and set $G = \mathbf{G}(\mathbb{A}_f)$ and Υ to be the set of all sufficiently small compact open subgroups of G (Definition B.5). If one chooses $\mathbb{Z}_p$-sheaves on these varieties and compatible isomorphisms between the pullbacks of these sheaves under Hecke actions (after choosing a suitable monoid and shrinking Υ if necessary), the degree j cohomology for any fixed j are then cohomology functors under pullbacks and trace maps of Hecke actions. Since the maps between Shimura varieties of two levels form covering maps of degree given by the index of the corresponding compact opens, they necessarily satisfy (Co) by [AGV73, Tome 3, Expose IX, §5]. If moreover, the group $\mathbf{G}$ satisfies axiom (SV5) of [Mil03], i.e. $\mathbf{Z}(\mathbb{Q})$ is discrete in $\mathbf{Z}(\mathbb{A}_f)$ where $\mathbf{Z} = \mathbf{Z}_{\mathbf{G}}$ is the center of $\mathbf{G}$, then these also satisfy axiom (M).

5.2. **Completions.** Let $M : (G, \Sigma, \Upsilon) \to R$-**Mod** be a cohomology functor. For any compact subgroup $C \subset \Sigma$ of G, we can define two completions

$$\overline{M}(C) = \varinjlim_{K \supset C} M(K), \quad M_{\mathrm{Iw}}(C) = \varprojlim_{K \supset C} M(K)$$

where the limits are taken with respect to the morphisms $\mathrm{pr}^\star$ and $\mathrm{pr}_\star$ respectively over all $K \in \Upsilon$ containing C. We call these the *inductive* and *Iwasawa* completions respectively. When $C = \{1\}$, we also denote the inductive completion by $\widehat{M}$. We denote by $j_K\colon M(K) \to \overline{M}(C)$ the natural map. If C is central (i.e. contained in the center of Σ), the space $\overline{M}(C)$ naturally carries a smooth action of Σ as follows. For an element $g \in \Sigma$ and $x \in \overline{M}(C)$, choose a compact open $K \in \Upsilon$ such that there exists $x_K \in M(K)$ with $j_K(x_K) = x$. By replacing K with a subgroup of $K \cap g^{-1}Kg$ contained in Υ, we may assume that $gKg^{-1} \in \Upsilon$. We then define $g \cdot x$ to be the image of x_K under the composition

$$M(K) \xrightarrow{[g]^\star} M(gKg^{-1}) \to \overline{M}(C).$$

It is a routine check that this is well-defined, and the action is smooth by property (C3) in Definition 5.1.

Lemma 5.4. *Suppose M is a Galois functor, C is central. Then $j_K : M(K) \xrightarrow{\sim} \overline{M}(C)^K$ for any $K \supset C$ and $K \in \Upsilon$. In particular, if we choose a left invariant Haar measure on G giving K measure one and $\Sigma = G$, then $M(K)$ is a (left) module over the Hecke algebra $\mathcal{H}(K\backslash G/K)$ with the action of the Hecke operator $\mathrm{ch}(K\sigma K)$ on $x \in M(K) \hookrightarrow \overline{M}(C)$ given by*

$$\mathrm{ch}(K\sigma K) \cdot x = \sum_{\alpha \in K\sigma K/K} \alpha \cdot j_K(x)$$

Proof. Let $x \in \overline{M}(C)^K$. Then x is in the image of the map j_L for some $L \in \Upsilon$ such that $L \triangleleft K$ and $L \supset C$. This implies that

$$x \in M(L)^{K/L} \xleftarrow{\sim} M(K)$$

by (G). The explicit action of the Hecke operator can then be read off via the integral in [BH19, Eqn 4.2.2]. □

5.3. **Hecke correspondences.** Let $M : (G, \Sigma, \Upsilon) \to R$-**Mod** be a cohomology functor. For every $K, K' \in \Upsilon$ and $\sigma \in \Sigma$, we have a diagram

$$\begin{array}{ccccc} & K \cap \sigma^{-1}K'\sigma & \xleftarrow{[\sigma]} & \sigma K \sigma^{-1} \cap K' & \\ \swarrow^{\mathrm{pr}} & & & & \searrow^{\mathrm{pr}} \\ K & & & & K' \end{array}$$

This induces a map

$$[K'\sigma K] : M(K) \xrightarrow{\mathrm{pr}^\star} M(K \cap \sigma^{-1}K'\sigma) \xrightarrow{[\sigma]^\star} M(\sigma K\sigma^{-1} \cap K') \xrightarrow{\mathrm{pr}_\star} M(K')$$

which we refer to as the *Hecke correspondence* induced by σ. It is straightforward to verify that $[K'\sigma K]$ only depends on the double coset $K'\sigma K$.

Lemma 5.5. *Suppose that $M : (G, \Sigma, \Upsilon) \to R$-**Mod** is Cartesian. Let $K, K', K'' \in \Upsilon$ and $\sigma, \tau \in \Sigma$.*

(a) If we write $K'\sigma K = \sqcup\, \alpha K$. then we have

$$j_{K'} \circ [K'\sigma K] = \sum \alpha \cdot j_K.$$

(b) The composition $j_{K''} \circ [K''\tau K'] \circ [K'\sigma K]$ is given by the convolution product of double cosets, i.e. if we write $K'\sigma K = \sqcup\, \alpha K$ and $K''\tau K' = \sqcup\, \beta K'$ then

$$j_{K''} \circ [K''\tau K'] \circ [K'\sigma K] = \sum (\beta\alpha) \cdot j_K.$$

(c) If M is moreover Galois, $\Sigma = G$ and a left invariant Haar measure on G is chosen giving K measure 1, then the actions on $M(K)$ of the correspondence $[K\sigma K]$ and the operator $\mathrm{ch}(K\sigma K)$ defined in the previous subsection agree.

Proof. Let $L' := \sigma K \sigma^{-1} \cap K'$ which is an element of Υ. We can (and do) choose a compact open subgroup $L \in \Upsilon$ satisfying $L \triangleleft K'$ and $L \subset L'$, and let $\{\gamma\}_{\gamma \in I}$ denote any set of representatives in K' of the coset space $K'/L' = L\backslash K'/L'$; for any such γ we set $L_\gamma = \gamma L' \gamma^{-1} \cap L = L$. By the Cartesian property, we obtain the following commutative diagram

$$\begin{array}{ccccc} & & \bigoplus_\gamma M(L) & \xrightarrow{\sum \mathrm{pr}_\star} & M(L) \\ & & \uparrow{\scriptstyle \sum[\gamma]^\star} & & \uparrow{\scriptstyle \mathrm{pr}^\star} \\ M(K) & \xrightarrow{[\sigma]^\star} & M(L') & \xrightarrow{\mathrm{pr}_\star} & M(K') \end{array} \qquad M(K) \xrightarrow{[K'\sigma K]} M(K')$$

which implies that $\mathrm{pr}^\star_{L,K'} \circ [K'\sigma K] = \sum_\gamma [\gamma\sigma]^\star$. It is then easily verified that $\gamma\sigma$ are distinct representatives of $K'\sigma K/K$, and that the action so defined is independent of the choice of representatives. This completes the proof of part (a). For (b), we note that the composition of the Hecke correspondences in the limit is given by

$$M(K) \xrightarrow{\sum[\alpha]^\star} M(L) \xrightarrow{\sum[\beta]^\star} M(J)$$

where $J \in \Upsilon$ is such that $J \triangleleft K''$ and $J \subset \tau L \tau^{-1}$. Part (c) is immediate. □

The next lemma is a useful criterion for when two Hecke correspondences commute with each other.

Lemma 5.6. *For $i = 1, 2$ set $\mathcal{P}_i = \mathcal{P}(G_i, \Sigma_i, \Upsilon_i)$ and $\mathcal{P} = \mathcal{P}(G, \Sigma, \Upsilon) = \mathcal{P}_1 \times \mathcal{P}_2$. Let $M : \mathcal{P} \to R$-**Mod** be a Cartesian cohomology functor, and take*
- $\sigma \in \Sigma_2$
- $J \in \Upsilon_2$
- $K = K_1 \times J$ *and* $L = L_1 \times J$ *both elements in* Υ
- $g = (g_1, 1) : L \to K$ *a morphism in* $\mathcal{P}$.

Then we have
$$[g]^\star \circ [K\sigma K] = [L\sigma L] \circ [g]^\star.$$
A similar result holds for pushforwards.

Proof. Let $K^g = gKg^{-1}$. Then we have $g(\sigma K \sigma^{-1} \cap K)g^{-1} = \sigma K^g \sigma^{-1} \cap K^g$. It suffices to prove that
$$\begin{array}{ccc} M(\sigma L \sigma^{-1} \cap L) & \xrightarrow{\mathrm{pr}_\star} & M(L) \\ {\scriptstyle \mathrm{pr}^\star}\uparrow & & \uparrow{\scriptstyle \mathrm{pr}^\star} \\ M(\sigma K^g \sigma^{-1} \cap K^g) & \xrightarrow{\mathrm{pr}_\star} & M(K^g) \end{array}$$
But this is immediate from the Cartesian property since
$$L \backslash K^g / (\sigma K^g \sigma^{-1} \cap K^g) = (L_1 \times J) \backslash (K_1^{g_1} \times J) / K_1^{g_1} \times (\sigma J \sigma^{-1} \cap J) = \{\mathrm{id}_K\}$$
and $L \cap (\sigma K^g \sigma^{-1} \cap K^g) = \sigma L \sigma^{-1} \cap L$. □

5.4. **Pushforwards between cohomology functors.** Let $\iota : H \hookrightarrow G$ be a closed subgroup, and suppose that M_H, M_G are cohomology functors for the triplets $(H, \Sigma_H, \Upsilon_H)$ and $(G, \Sigma_G, \Upsilon_G)$ respectively. We require $\iota(\Sigma_H) \subset \Sigma_G$ and that for all $U \in \Upsilon_H$, $K \in \Upsilon_G$, we have $U \cap K \in \Upsilon_H$. We say that an element $(U, K) \in \Upsilon_H \times \Upsilon_G$ forms a *compatible* pair if $U \subset K$. A *morphism* of compatible pairs $(V, L) \to (U, K)$ is a pair of morphisms $[h] : V \to U$, $[h] : L \to K$ for $h \in \Sigma_H$.

Definition 5.7. A *pushforward* $\iota_\star : M_H \to M_G$ is a collection of maps $M_H(U) \to M_G(K)$ for all compatible pairs (U, K), which commute with the pushforward maps induced by morphisms of compatible pairs coming from Σ_H^{-1}. A *Cartesian pushforward* $\iota_\star : M_H \to M_G$ is a pushforward such that for all $U \in \Upsilon_H$, $L, K \in \Upsilon_G$ and $U, L \subset K$, we have a commutative diagram
$$\begin{array}{ccc} \bigoplus_\gamma M_H(U_\gamma) & \xrightarrow{\sum [\gamma]_\star} & M_G(L) \\ {\scriptstyle \sum \mathrm{pr}^\star}\uparrow & & \uparrow{\scriptstyle \mathrm{pr}^\star} \\ M_H(U) & \xrightarrow{\iota_\star} & M_G(K) \end{array}$$
where $\gamma \in U \backslash K / L$ is a fixed set of representatives, $U_\gamma = \gamma L \gamma^{-1} \cap U$ and $[\gamma]_\star : M_H(U_\gamma) \to M_G(L)$ is the composition $M_H(U_\gamma) \to M_G(\gamma L \gamma^{-1}) \to M_G(L)$.

Remark 5.8. When $G = H$ and $\iota_\star = \mathrm{pr}_\star$, one recovers the conditions for M_G to be Cartesian. We point out that what we refer to as a Cartesian pushforward here is simply called a pushforwad in [Loe19].

Assume for the rest of this section that H is unimodular, $\mathcal{P}(H, \Sigma_H, \Upsilon_H)$, $\mathcal{P}(G, \Sigma_G, \Upsilon_G)$ are categories of compact opens with $\Sigma_H = H$, $\Sigma_G = G$ and R is a $\mathbb{Q}$-algebra. We fix left invariant Haar measures dh, dg on H, G respectively that take values in $\mathbb{Q}$ on the respective compact opens, and denote by $\mathrm{Vol}(-)$ the volume of the corresponding compact opens. We consider $\mathcal{H}(G, R)$ as an algebra under the usual convolution product $*$ (see [BH19, §1.4.1]) and a left representation of G with $g \in G$ acting on $\xi \in \mathcal{H}(G, R)$ via $\rho_g(\xi) : x \mapsto \xi(xg)$. The action ρ of G induces an action of the Hecke algebra on itself [BH19, 1.4.2], which we denote by $\xi *_\rho \zeta$ for $\xi \in \mathcal{H}(G, R)$ (considered as a ring element) and $\zeta \in \mathcal{H}(G, R)$ (considered as a module element). Finally, for $\xi \in \mathcal{H}(G, R)$, we let ξ^t denote its transpose $x \mapsto \xi(x^{-1})$.

Definition 5.9. Given smooth representations τ of H, σ of G and C a central compact subgroup of G which acts trivially on σ, we consider $\tau \otimes \mathcal{H}(G, R)^C$ and σ smooth representations of $H \times G$ by the following *extended action.*

- $(h,g) \in H \times G$ acts on $x \otimes \xi \in \tau \otimes \mathcal{H}(G,R)^C$ via $x \otimes \xi \mapsto hx \otimes \xi(h^{-1}(-)g)$.
- $H \times G$ acts on σ via projection to G which acts via its natural action.

An *intertwining map* $\mathfrak{Z} : \tau \otimes \mathcal{H}(G,R)^C \to \sigma$ is just a morphism of $H \times G$ representations.

Lemma 5.10. *Let τ, σ and $\mathfrak{Z}$ be as above.*

(a) For all $x \in \tau$, $\xi_1, \xi_2 \in \mathcal{H}(G,R)^C$,

$$\mathfrak{Z}(x \otimes \xi_1 * \xi_2) = \xi_2^t \cdot \mathfrak{Z}(x \otimes \xi_1).$$

(b) (Frobenius Reciprocity) Let $\sigma^\vee$ denote the smooth dual of σ and denote by $\langle \cdot, \cdot \rangle : \sigma^\vee \times \sigma \to R$ the induced pairing. Consider $\tau \otimes \sigma$ as a smooth H-representation via $h \cdot (x \otimes \varphi) = h \cdot x \otimes \iota(h)\varphi$ for $h \in H$, $x \in \tau$, $\varphi \in \sigma^\vee$. There is a unique element $\mathfrak{z} \in \mathrm{Hom}_H(\tau \otimes \sigma^\vee, R)$ (depending on the choice of dg) such that

$$\langle \varphi, \mathfrak{Z}(x \otimes \xi) \rangle = \mathfrak{z}(x \otimes \xi \cdot \varphi)$$

for all $\varphi \in \sigma^\vee$, $x \in \tau$, $\xi \in \mathcal{H}(G,R)^C$.

Proof. For (a), note that the G-equivariance of $\mathfrak{Z}$ tells us that $\mathfrak{Z}(x \otimes (\xi_2^t *_\rho \xi_1)) = \xi_2^t \cdot \mathfrak{Z}(x \otimes \xi_1)$ [BH19, Proposition 1, p.35]. Since $\xi_1 * \xi_2 = \xi_2^t *_\rho \xi_1$, we have the first claim.

(b) Given $x \in \tau$, $\varphi \in \sigma^\vee$, choose a compact open $K \supset C$ such that the idempotent $e_K = \mathrm{Vol}(K; dg)^{-1}\mathrm{ch}(K) \in \mathcal{H}(G,R)^C$ fixes φ, and set $\mathfrak{z}(x \otimes \varphi) := \langle \varphi, \mathfrak{Z}(x \otimes e_K) \rangle$. It is easily verified that $\mathfrak{z}$ is well-defined, satisfies the property claimed above and is uniquely determined by its properties. □

We now construct a "completed pushforward" from a given pushforward $M_H \to M_G$ i.e. an intertwining map of $H \times G$ representations as above. This is a straightforward generalisation of the "Lemma–Eisenstein" map in [LSZ17, Lemma 8.2.1].

Proposition 5.11 (Completed Pushforward)**.** *Suppose M_H, M_G are cohomology functors with M_H satisfying (Co), M_G satisfying (M). Consider $\widehat{M}_H \otimes \mathcal{H}(G,R)^C$ and $\overline{M}_G(C)$ as smooth $H \times G$ representations via the extended action. Then for any pushforward $\iota_\star : M_H \to M_G$, there is a unique (depending on the choice of dh) intertwining map of $H \times G$ representations*

$$\hat{\iota}_\star : \widehat{M}_H \otimes \mathcal{H}(G,R)^C \to \overline{M}_G(C)$$

satisfying the following compatibility condition: for all compatible pairs $(U,K) \in \Upsilon_H \times \Upsilon_G$ with $K \supset C$, $x \in M_H(U)$, $y \in M_G(K)$ such that $\iota_\star(x) = y$, we have $\hat{\iota}_\star(j_U(x) \otimes \mathrm{ch}(K)) = \mathrm{Vol}(U) \cdot j_K(y)$.

Proof. We first assume that $C = \{1\}$. Elements of $\widehat{M}_H \otimes \mathcal{H}(G,R)$ are spanned by pure tensors of the form $x \otimes \mathrm{ch}(gK)$ for $x \in \widehat{M}_H$, $g \in G$ and $K \in \Upsilon_G$. Indeed, for any compact open subgroup K of G, there exists a compact open subgroup such that $K' \subset K$ and $K' \in \Upsilon$, because we require that Υ contains a basis of the identity. We will define the map on these pure tensors and extend linearly.

Choose a compact open subgroup $U \in \Upsilon_H$ such that U fixes x, $U \subset gKg^{-1} \cap H$, and let $x_U \in M_H(U)$ be an element that maps to x under j_U. Then, we define $\hat{\iota}_\star(x \otimes \mathrm{ch}(gK))$ to be the image of $\mathrm{Vol}(U)x_U$ under the composition

$$M_H(U) \xrightarrow{\iota_\star} M_G(gKg^{-1}) \xrightarrow{[g]_\star} M_G(K) \xrightarrow{j_K} \widehat{M}_G$$

To see that this is independent of choice of x_U (and U), assume that U is replaced by another open compact subgroup V and x_U by $x_V \in M_H(V)$. We can then choose an even smaller compact open $V' \in \Upsilon_H$, $V' \subset U \cap V$ such that $x_U, x_V \mapsto x_{V'}$ under pullbacks, and we may therefore assume that $V \subset U$, and $x_U \mapsto x_V$ under $\mathrm{pr}^*_{V,U}$. Since M_H is a covering functor, $x_V \mapsto [U:V]x_U$ under $\mathrm{pr}_{V,U,*}$ Hence both $\mathrm{Vol}(U)x_U$ and $\mathrm{Vol}(V)x_V$ map to the same element in $\widehat{M}_G$.

It remains to show that that the various relations among elements of $\mathcal{H}(G,R)$ do not give conflicting images on the sums of these simple tensors. To this end, let $L, K \in \Upsilon_G$ such that L is a normal subgroup of K. We want to verify that

$$\hat{\iota}_\star(x \otimes \mathrm{ch}(K)) = \sum_{\gamma \in K/L} \hat{\iota}_\star(x \otimes \mathrm{ch}(\gamma L)).$$

The general case for two different representations of $\xi \in \mathcal{H}(G,R)$ as sums of characteristic functions of left cosets can be reduced to this case by successively choosing normal subgroup for pairs and twisting by the action of G, by establishing G-equivariance of such a (a priori hypothetical) map first.

Choose $U \in \Upsilon_H$ such that U fixes x and $U \subset L \cap H$. Note that as $L \triangleleft K$, $\gamma L \gamma^{-1} \cap H = L \cap H$ for any $\gamma \in K$. As before, let $x_U \in M_H(U)$ be an element mapping to x. By definition

$$\hat{\iota}_\star(x \otimes \mathrm{ch}(K)) = j_K \circ \iota_{U,K,\star}(\mathrm{Vol}(U)x_U), \qquad \hat{\iota}_\star(x \otimes \mathrm{ch}(\gamma L)) = j_L \circ [\gamma]_\star \circ \iota_{U,L,\star}(\mathrm{Vol}(U)x_U).$$

As $j_K = j_L \circ \mathrm{pr}^\star_{L,K}$, it suffices to prove that

$$\mathrm{pr}^\star_{L,K} \circ \iota_{U,K,\star} = \sum_\gamma [\gamma]_\star \circ \iota_{U,L,\star} = \sum_\gamma [\gamma]^\star \circ \iota_{U,L,\star}.$$

as maps $M_H(U) \to M_G(L)$, where the last equality follows from the fact that L is normal in K, so we can replace the set of representatives by their inverses. This equality is then justified by replacing $\iota_{U,K,\star} = \mathrm{pr}_{L,K,\star} \circ \iota_{U,L,\star}$ and invoking the axiom (M) for M_G. Therefore, $\hat{\iota}_\star$ is well-defined.

We now check that $\hat{\iota}_\star$ is $H \times G$ equivariant with the said actions: let

$$(h,g) \in H \times G, \qquad v = x \otimes \mathrm{ch}(g_1 K) \in \widehat{M}_H \otimes \mathcal{H}(G,R).$$

Then,

$$(h,g) \cdot v = hx \otimes \mathrm{ch}(hg'K')$$

where $g' = g_1 g^{-1}$, $K' = gKg^{-1}$. Choose $U \in \Upsilon_H$ such that U fixes x, $U \subset g_1 K g_1^{-1} \cap H$ and $x_U \in M_H(U)$ that maps to x. Then, $hUh^{-1} \subset hg_1K(hg_1)^{-1} \cap H = hg'K'(hg')^{-1} \cap H$, and $[h]^\star x_U \in M_H(hUh^{-1})$ maps to hx. Since $[h]^\star = [h^{-1}]_\star$, we obtain a commutative diagram

$$\begin{array}{ccc} M_H(U) & \xrightarrow{\iota_\star} & M_G(g_1 K g_1^{-1}) \\ {\scriptstyle [h]^\star}\downarrow & & \downarrow{\scriptstyle [h]^\star} \\ M_H(hUh^{-1}) & \xrightarrow{\iota_\star} & M_G(hg'K'(hg')^{-1}) \end{array}$$

Now,

$$(h,g) \cdot \hat{\iota}_\star(v) = \mathrm{Vol}(U) \cdot [g \cdot (j_K \circ [g_1]_\star \circ \iota_\star(x_U))]$$
$$\hat{\iota}_\star((h,g) \cdot v) = \mathrm{Vol}(hUh^{-1}) \cdot (j_{K'} \circ [hg']_\star \circ \iota_\star([h]^\star x_U)).$$

As H is unimodular, $\mathrm{Vol}(U) = \mathrm{Vol}(hUh^{-1})$ and it therefore suffices to verify that $[g^{-1}]_\star \circ [g_1]_\star \circ \iota_\star(x_U) = [hg']_\star \circ \iota_\star([h]^\star x_U)$ as elements of $M_G(K')$. But this follows immediately from the commutativity of the above diagram and the composition law $[hg]_\star = [g]_\star[h]_\star$. This finishes the checking of equivariance. The statement involving C is obtained by taking C invariants on both sides, and noticing that the map factors through $\overline{M}(C) \subset \widehat{M}^C$. It is easily seen that this map is uniquely determined with the prescribed actions of $H \times G$ and the compatibility condition with $\iota_\star$. ☐

Ideally one would like an integral version of the above proposition but such a map doesn't exist in general due to the presence of the volume factors in the definition. It is however possible to relate the pushforward map on finite level to an integral version, as follows.

Let $\mathcal{O}$ be an integral domain with field of fractions Φ, and let $M_\mathcal{O}$ and M_Φ be cohomology functors for (G,Σ,Υ) valued in $\mathcal{O}$ and Φ respectively. We say that $M_\mathcal{O}$ and M_Φ are *compatible under base change* if there exists a collection of $\mathcal{O}$-linear maps $M_\mathcal{O}(K) \to M_\Phi(K)$ which are natural in K and commute with pullbacks and pushforwards.

Let $\iota\colon H \hookrightarrow G$ be as above, and let $(M_{H,\mathcal{O}}, M_{H,\Phi})$ and $(M_{G,\mathcal{O}}, M_{G,\Phi})$ be pairs of compatible cohomology functors. Suppose that we have a pushforward $\iota_\star\colon M_{H,\Phi} \to M_{G,\Phi}$ that restricts to a pushforward $M_{H,\mathcal{O}} \to M_{G,\mathcal{O}}$. Then we obtain the following corollary.

Corollary 5.12. *Let $\Sigma_G = G$ and $\Sigma_H = H$. Keeping with the same notation as above, let $U \in \mathcal{P}(H,\Sigma_H,\Upsilon_H)$ and $K \in \mathcal{P}(G,\Sigma_G,\Upsilon_G)$ be compact open subgroups, and suppose that we have an element $g \in G$ such that*

$U \subset gKg^{-1} \cap H$. Then we have a commutative diagram:

$$\begin{array}{ccccc} \widehat{M}_{H,\Phi} & & \xrightarrow{x \mapsto \hat{\iota}_*(x \otimes \mathrm{ch}(gK))} & & \widehat{M}_{G,\Phi} \\ j_U \uparrow & & & & \uparrow \mathrm{Vol}(U) \cdot j_K \\ M_{H,\mathcal{O}}(U) & \xrightarrow{\iota_*} & M_{G,\mathcal{O}}(gKg^{-1}) & \xrightarrow{[g]_*} & M_{G,\mathcal{O}}(K) \end{array}$$

6. Vertical norm relations at split primes

In this section we construct classes in the "p-direction" following the general method outlined in [Loe19]. As a result we will obtain cohomologically trivial cycles which are compatible as one goes up in the *anti-cyclotomic* tower. We will also need results of this section in the construction of the Euler system map. We do not require n to be odd in this section.

6.1. **Group theoretic set-up.** Let p be an odd prime which splits in $E/\mathbb{Q}$. Then we can consider the local versions of the groups in section 2.1 at the prime p – we will denote these groups by unscripted letters. We have identifications

$$\begin{array}{rcl} G = \mathbf{G}_{\mathbb{Q}_p} & \cong & \mathrm{GL}_1 \times \mathrm{GL}_{2n} \\ H = \mathbf{H}_{\mathbb{Q}_p} & \cong & \mathrm{GL}_1 \times \mathrm{GL}_n \times \mathrm{GL}_n \\ T = \mathbf{T}_{\mathbb{Q}_p} & \cong & \mathrm{GL}_1 \end{array}$$

Since all of these groups are unramified, they all admit a reductive model over $\mathbb{Z}_p$ which we fix and, by abuse of notation, denote by the same letters.

Definition 6.1. Let Q denote the Siegel parabolic subgroup of G given by $Q = \mathrm{GL}_1 \times \begin{pmatrix} * & * \\ & * \end{pmatrix}$ and let $J \subset G(\mathbb{Z}_p)$ denote the parahoric subgroup associated with Q (i.e. all elements in $G(\mathbb{Z}_p)$ which lie in Q modulo p). We identify the Levi of Q with the subgroup $\iota(H)$.

The automorphic representations that we will consider will be "ordinary" with respect to a certain Hecke operator associated with the parabolic subgroup $Q \times T \subset \widetilde{G}$, which we will now define. Let τ and $\widetilde{\tau}$ be the elements

$$\tau = \left(1, \begin{pmatrix} p & \\ & 1 \end{pmatrix}\right) \in G(\mathbb{Q}_p) \qquad \widetilde{\tau} = \left(1, \begin{pmatrix} p & \\ & 1 \end{pmatrix}\right) \times 1 \in \widetilde{G}(\mathbb{Q}_p).$$

Then for any compact open subgroup $\widetilde{K} \subset J \times T$, we let $\mathcal{T}$ denote the Hecke operator associated with the double coset $[\widetilde{K}\widetilde{\tau}^{-1}\widetilde{K}]$. More concretely, $\mathcal{T}$ is the operator associated with the following correspondence:

$$\begin{array}{ccccc} & \mathrm{Sh}_{\widetilde{\mathbf{G}}}(\widetilde{K} \cap \widetilde{\tau}\widetilde{K}\widetilde{\tau}^{-1}) & \xleftarrow{[\widetilde{\tau}^{-1}]} & \mathrm{Sh}_{\widetilde{\mathbf{G}}}(\widetilde{K} \cap \widetilde{\tau}^{-1}\widetilde{K}\widetilde{\tau}) & \\ \swarrow & & & & \searrow \\ \mathrm{Sh}_{\widetilde{\mathbf{G}}}(\widetilde{K}) & & & & \mathrm{Sh}_{\widetilde{\mathbf{G}}}(\widetilde{K}) \end{array}$$

Explicitly, if p_1 and p_2 denote the left-hand and right-hand projection maps respectively, then $\mathcal{T} = p_{2,*} \circ [\widetilde{\tau}^{-1}]_* \circ p_1^*$ (see also section 5.3). In all choices $\widetilde{K}' \subset \widetilde{K}$ that we will consider, this operator will be compatible with the pushforward morphism $(\mathrm{pr}_{\widetilde{K}',\widetilde{K}})_*$, where $\mathrm{pr}_{\widetilde{K}',\widetilde{K}} \colon \mathrm{Sh}_{\widetilde{\mathbf{G}}}(\widetilde{K}') \to \mathrm{Sh}_{\widetilde{\mathbf{G}}}(\widetilde{K})$ is the natural map; so we will often omit the level when referring to it (see [Loe19, §4.5]). Note that if $\widetilde{K} = K \times C$ is decomposible, then $\mathcal{T}$ only acts on the cohomology of the first factor of $\mathrm{Sh}_{\widetilde{\mathbf{G}}}(\widetilde{K}) \cong \mathrm{Sh}_{\mathbf{G}}(K) \times \mathrm{Sh}_{\mathbf{T}}(C)$. We will denote the Hecke correspondence induced on $\mathrm{Sh}_G(K)$ also by $\mathcal{T}$.

6.2. **The flag variety.** We now consider the spherical variety that underlies the whole construction in this section. Let $\overline{Q}$ denote the opposite of Q (with respect to the standard torus) with unipotent radical $\overline{N}$. We will also abusively write $\overline{N}$ for the unipotent radical of $\overline{Q} \times T$. Then we consider the flag variety $\mathcal{F} = \widetilde{G}/(\overline{Q} \times T) = G/\overline{Q}$ over $\mathrm{Spec}\,\mathbb{Z}_p$, which has a natural left action of $\widetilde{G}$ (which factors through the projection to G). Let $\widetilde{u} \in \widetilde{G}(\mathbb{Z}_p)$ denote the element

$$\widetilde{u} = \left(1, \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}\right) \times 1$$

and consider the following reductive subgroup $L^\circ \lhd \iota(H) \times T$ given by

$$L^\circ = \mathrm{GL}_1 \times \begin{pmatrix} X & \\ & Y \end{pmatrix} \times 1.$$

Then we verify the following properties.

Lemma 6.2. *Keeping the same notation as above, the following are satisfied:*

(1) The stabiliser $\mathrm{Stab}_H([\widetilde{u}]) = (\iota, \nu)(H) \cap \widetilde{u}(\overline{Q} \times T)\widetilde{u}^{-1}$ *is contained in* $\widetilde{u}\left(\overline{N} \cdot L^\circ\right)\widetilde{u}^{-1}$.
(2) The H-orbit of $\widetilde{u}$ in $\mathcal{F}$ (over $\operatorname{Spec} \mathbb{Z}_p$*) is Zariski open.*

Proof. The first part is a routine check that is left to the reader. For the second part we note that we have the following equality

$$(\iota, \nu)(H) \cap \widetilde{u}(\overline{Q} \times T)\widetilde{u}^{-1} = \left\{ \mathrm{GL}_1 \times \begin{pmatrix} X & \\ & X \end{pmatrix} \times 1 : X \in \mathrm{GL}_n \right\}$$

which implies that the orbit of $[\widetilde{u}]$ can be identified with a copy of $\mathrm{GL}_n \subset \mathcal{F}$. One can check that this is open. □

Remark 6.3. In the notation of [Loe19, §4.4, §4.6], we have

- $Q_H^\circ = H$
- $\overline{Q}_{\widetilde{G}} = \overline{Q} \times T$ with unipotent radical $\overline{N}_{\widetilde{G}} = \overline{N}$ and Levi $L_{\widetilde{G}} = \iota(H) \times T$.
- $L_{\widetilde{G}}^\circ = L^\circ$ and $\overline{Q}_{\widetilde{G}}^\circ = \overline{N} \cdot L^\circ$.

Proceeding as in [Loe19, §4.4], we define the following level subgroups

- $U_r = \{g \in \widetilde{G}(\mathbb{Z}_p) : g \pmod{p^r} \in \overline{N} \cdot L^\circ \text{ and } \widetilde{\tau}^{-r} g \widetilde{\tau}^r \in \widetilde{G}(\mathbb{Z}_p)\}$
- $V_r = \widetilde{\tau}^{-r} U_r \widetilde{\tau}^r$

for $r \geq 0$. We let $D_{p^r} \subset T(\mathbb{Z}_p)$ denote the group of all elements $x \equiv 1$ modulo p^r. Then V_r is a compact open subgroup of $J \times D_{p^r}$ for $r \geq 1$.

Lemma 6.4. *Let $r \geq 1$ and set $V_r' = V_r \cap \widetilde{\tau} V_r \widetilde{\tau}^{-1}$. Then*

(a) $V_r' \backslash V_r / V_{r+1}$ is a singleton and $V_r' \cap V_{r+1} = V_{r+1}'$.
(b) Set $(J \times D_{p^r})' = \widetilde{\tau}(J \times D_{p^r})\widetilde{\tau}^{-1} \cap (J \times D_{p^r})$. The double coset space

$$(J \times D_{p^r})' \backslash J \times D_{p^r} / V_r$$

is a singleton and $V_r \cap (J \times D_{p^r})' = V_r'$.

Proof. In terms of U_r, the first claim in part (a) amounts to showing that $(U_r \cap \widetilde{\tau} U_r \widetilde{\tau}^{-1}) \backslash U_r / \widetilde{\tau}^{-1} U_{r+1} \widetilde{\tau}$ is a singleton. Notice that we have the Iwahori decomposition

$$U_r = \overline{N}_0 \cdot L_r \cdot N_r$$

where N, $\overline{N}$ are the unipotent radicals of $Q \times T$, $\overline{Q} \times T$ respectively, $L = \iota(H) \times T \subset G \times T$ is the Levi, and $N_r := \widetilde{\tau}^r N(\mathbb{Z}_p) \widetilde{\tau}^{-r}$, $\overline{N}_r := \widetilde{\tau}^{-r} \overline{N}(\mathbb{Z}_p) \widetilde{\tau}^r$ and $L_r = L^\circ(\mathbb{Z}_p)(1 \times D_{p^r}) \subset G(\mathbb{Z}_p) \times T(\mathbb{Z}_p)$. One then easily sees that

$$U_r \cap \widetilde{\tau} U_r \widetilde{\tau}^{-1} = \overline{N}_0 \cdot L_r \cdot N_{r+1}, \qquad \widetilde{\tau}^{-1} U_{r+1} \widetilde{\tau} = \overline{N}_1 \cdot L_{r+1} \cdot N_r$$

which implies that $U_r = \left(U_r \cap \widetilde{\tau} U_r \widetilde{\tau}^{-1}\right) \cdot \left(\widetilde{\tau}^{-1} U_{r+1} \widetilde{\tau}\right)$ as required. As

$$V_r' = \overline{N}_r \cdot L_r \cdot N_1, \qquad V_{r+1} = \overline{N}_{r+1} \cdot L_{r+1} \cdot N_0$$

their intersection equals $\overline{N}_{r+1} \cdot L_{r+1} \cdot N_1 = V_{r+1}'$. The proof of part (b) is similar. □

6.3. **Integral sheaves.** In this section we introduce the integral versions of the sheaves constructed in sections 2.5–2.6 and discuss the corresponding branching law. Unfortunately, just choosing a $\mathbb{Z}_p$-lattice inside the representation is not sufficient in general, since there does not exist a lattice inside the self-dual representation V which is stable under both $G(\mathbb{Z}_p)$ and τ^{-1} (unless V is the trivial representation). Therefore we must consider a modified action on the lattice – essentially, this amounts to renormalising the $\mathcal{T}$ operator so that the action on the cohomology of $\mathscr{V}$ is integral. The construction is the same as that in [DJR18, §1.2], however we are using opposite conventions to stay consistent with the construction in [Loe19].

We continue to use the notation introduced in sections 6.1–6.2. Let V be an algebraic representation of $G = \mathbf{G}_{\mathbb{Q}_p}$ that is self-dual of weight $\mathbf{c} = (0; c_1, \ldots, c_{2n})$ as in Definition 2.12. We fix once and for all a highest weight vector $v_{\mathbf{c}} \in V$, so that the action of the standard diagonal torus inside G is given by

$$\mathrm{diag}(a_0; a_1, \ldots, a_{2n}) \cdot v_{\mathbf{c}} = \mu(a_0; a_1, \ldots, a_{2n}) v_{\mathbf{c}}$$

where μ is the character $\mathrm{diag}(a_0; a_1, \ldots, a_{2n}) \mapsto a_1^{c_1} \cdots a_{2n}^{c_{2n}}$. This vector can also be viewed as a highest weight vector in $\widetilde{V} := V \boxtimes \mathbf{1}$ – in this case, we denote it by $\widetilde{v}_{\mathbf{c}}$.

Definition 6.5. Let T denote the $G(\mathbb{Z}_p)$-stable lattice

$$T = \mathcal{U}(\overline{\mathfrak{u}}_{\mathbb{Z}_p}) \cdot v_{\mathbf{c}}$$

inside V, where $\mathcal{U}(\overline{\mathfrak{u}}_{\mathbb{Z}_p})$ is the universal enveloping algebra of the subgroup

$$\overline{U} = \{1\} \times \left\{ \begin{array}{c} \text{lower triangular} \\ \text{matrices} \end{array} \right\}.$$

This extends trivially to a $\widetilde{G}(\mathbb{Z}_p)$-stable lattice $\widetilde{T} \subset \widetilde{V}$

Let τ_0 denote the following $(2n \times 2n)$-block matrix

$$\tau_0 := \begin{pmatrix} p & \\ & 1 \end{pmatrix} \in G_0(\mathbb{Q}_p).$$

Let $J_0 \subset G_0(\mathbb{Z}_p)$ denote the parahoric subgroup of elements which land in Q_0 modulo p, and let $J := \mathbb{Z}_p^\times \times J_0 \subset G(\mathbb{Z}_p)$ and $\widetilde{J} := J \times \mathbb{Z}_p^\times \subset \widetilde{G}(\mathbb{Z}_p)$. We introduce the following monoids and actions on the lattices introduced in Definition 6.5.

Lemma 6.6. *Consider the following sets:*

- $\Sigma_0 := J_0 \cdot \{\tau_0^{-r} : r \in \mathbb{Z}_{\geq 0}\} \cdot J_0$
- $\Sigma := \mathbb{Q}_p^\times \times \Sigma_0$ *as a subset of* $G(\mathbb{Q}_p)$
- $\widetilde{\Sigma} := \Sigma \times T(\mathbb{Q}_p)$ *as a subset of* $\widetilde{G}(\mathbb{Q}_p)$.

Then Σ *and* $\widetilde{\Sigma}$ *are open submonoids of* $G(\mathbb{Q}_p)$ *and* $\widetilde{G}(\mathbb{Q}_p)$ *respectively, and the actions of* J *and* $\widetilde{J}$ *extend to actions* $\bullet$ *of* Σ *and* $\widetilde{\Sigma}$ *on* T *and* $\widetilde{T}$ *which factor through* Σ_0 *and satisfy:*

$$\tau^{-1} \bullet v = \mu(\tau)(\tau^{-1} \cdot v) \qquad \widetilde{\tau}^{-1} \bullet \widetilde{v} = \mu(\tau)(\widetilde{\tau}^{-1} \cdot v)$$

for all $v \in T$ *and* $\widetilde{v} \in \widetilde{T}$.

Proof. Recall from the proof of Lemma 6.4 that we have the Iwahori decompostion

$$J = \overline{N}_1 \cdot H(\mathbb{Z}_p) \cdot N(\mathbb{Z}_p) = N(\mathbb{Z}_p) \cdot H(\mathbb{Z}_p) \cdot \overline{N}_1$$

where $N \subset Q$ is the unipotent radical and $\overline{N}_r = \tau^{-r} \overline{N}(\mathbb{Z}_p) \tau^r$. The result now follows from the fact that $\tau^{-1} \overline{Q} \tau \subset \overline{Q}$ and $\tau N \tau^{-1} \subset N$, which allows us to extend the action of J by letting τ^{-1} act trivially on the highest weight vector (c.f. also [DJR18, §1.2]). The statement for $\widetilde{G}$ is similar. □

Using the above action, we can now define equivariant sheaves for the integral lattices T and $\widetilde{T}$ which are compatible with the rational versions by relaxing the condition that $\sigma^\star \mathscr{T}_K \to \mathscr{T}_L$ are isomorphisms. More precisely, we let $\mathscr{T}$ correspond to the collection of the sheaves $\mathscr{T}_K$ with morphisms

$$\sigma^\star \mathscr{T}_K \xrightarrow{\sigma \bullet} \mathscr{T}_L$$

induced from the $\bullet$-action, for all $\sigma \in \Sigma$ satisfying $\sigma^{-1}L\sigma \subset K$. In particular, we can still define pullbacks and pushforwards on the cohomology of these sheaves as follows. Let $\sigma \in \Sigma$ and $K, L \subset G(\mathbb{Q}_p)$ compact open subgroups satisfying $\sigma^{-1}L\sigma \subset K$. Then we define the pullback morphism as the composition

$$[\sigma]^\star\colon \mathrm{H}^i_{\text{ét}}\left(\mathrm{Sh}_{\mathbf{G}}(K), \mathscr{T}_K(j)\right) \to \mathrm{H}^i_{\text{ét}}\left(\mathrm{Sh}_{\mathbf{G}}(L), \sigma^\star \mathscr{T}_K(j)\right) \to \mathrm{H}^i_{\text{ét}}\left(\mathrm{Sh}_{\mathbf{G}}(L), \mathscr{T}_L(j)\right).$$

Similarly if we take $\sigma \in \Sigma^{-1}$ and $K, L \subset G(\mathbb{Q}_p)$ compact open subgroups satisfying $\sigma^{-1}L\sigma \subset K$, then we can define the corresponding pushforward morphism as the composition

$$\begin{aligned}[\sigma]_\star\colon \mathrm{H}^i_{\text{ét}}\left(\mathrm{Sh}_{\mathbf{G}}(L), \mathscr{T}_L(j)\right) &\xrightarrow{[\sigma^{-1}]^\star} \mathrm{H}^i_{\text{ét}}\left(\mathrm{Sh}_{\mathbf{G}}(\sigma^{-1}L\sigma), \mathscr{T}_{\sigma^{-1}L\sigma}(j)\right) \\ &\xrightarrow{\sim} \mathrm{H}^i_{\text{ét}}\left(\mathrm{Sh}_{\mathbf{G}}(K), \mathrm{pr}_! \,\mathrm{pr}^! \,\mathscr{T}_K(j)\right) \\ &\to \mathrm{H}^i_{\text{ét}}\left(\mathrm{Sh}_{\mathbf{G}}(K), \mathscr{T}_K(j)\right)\end{aligned}$$

where the middle isomorphism is induced from the pushforward along $\mathrm{pr}\colon \mathrm{Sh}_{\mathbf{G}}(\sigma^{-1}L\sigma) \to \mathrm{Sh}_{\mathbf{G}}(K)$ and the fact that we have $\mathrm{pr}^! \mathscr{T}_K = \mathrm{pr}^* \mathscr{T}_K \cong \mathscr{T}_{\sigma^{-1}L\sigma}$, and the last map is the trace map. We have similar maps for the sheaves corresponding to the group $\widetilde{\mathbf{G}}$.

Lemma 6.7. *The natural maps*

$$\mathrm{H}^i_{\text{ét}}\left(\mathrm{Sh}_{\mathbf{G}}, \mathscr{T}(j)\right) \to \mathrm{H}^i_{\text{ét}}\left(\mathrm{Sh}_{\mathbf{G}}, \mathscr{V}(j)\right) \qquad \mathrm{H}^i_{\text{ét}}\left(\mathrm{Sh}_{\widetilde{\mathbf{G}}}, \widetilde{\mathscr{T}}(j)\right) \to \mathrm{H}^i_{\text{ét}}\left(\mathrm{Sh}_{\widetilde{\mathbf{G}}}, \widetilde{\mathscr{V}}(j)\right)$$

are equivariant for the actions of $\mathbf{G}(\mathbb{A}_f^p) \times J$ and $\widetilde{\mathbf{G}}(\mathbb{A}_f^p) \times \widetilde{J}$ respectively. Furthermore, the action of $\mathcal{T}$ on the left-hand side is intertwined with the action of $\mathcal{U}'_S := \mu(\tau)\mathcal{T}$ on the right-hand side.

Proof. The first part follows from the fact that $T \subset V$ and $\widetilde{T} \subset \widetilde{V}$ are equivariant embeddings for the actions of J and $\widetilde{J}$ respectively. The last assertion follows from the fact that $\widetilde{\tau}^{-1} \bullet \widetilde{v} = \mu(\tau)(\widetilde{\tau}^{-1} \cdot \widetilde{v})$ for all $\widetilde{v} \in \widetilde{T}$ (recall that $\mathcal{T}$ is the Hecke operator $[\widetilde{K}\widetilde{\tau}^{-1}\widetilde{K}]$). □

Since we have slightly modified the action of Σ we have to modify the branching law discussed in section 2.6 when working integrally. Let $\lambda = \sum_{i=1}^n c_i$ and consider the following character

$$\begin{aligned}|\det|_p^{\frac{\lambda}{n}} : \Sigma \cap H(\mathbb{Q}_p) = \widetilde{\Sigma} \cap H(\mathbb{Q}_p) \quad &\to \quad \mathbb{Q}_{>0}^\times \subset \mathbb{Q}_p^\times \\ (\alpha; h_1, h_2) \quad &\mapsto \quad |\det h_1 \cdot \det h_2|_p^{\frac{\lambda}{n}}\end{aligned}$$

and let $\mathbb{Z}_p[\lambda/n]$ denote the trivial equivariant sheaf on $\mathrm{Sh}_{\mathbf{H}}$ whose action of $\Sigma \cap H(\mathbb{Q}_p)$ is twisted by the character $|\det|_p^{\lambda/n}$. Since the representation $\widetilde{V}$ is self-dual, we can fix a $H(\mathbb{Q}_p)$-equivariant embedding $\mathbb{Q}_p \hookrightarrow \widetilde{V}$ which restricts to a $H(\mathbb{Z}_p)$-equivariant embedding $\mathbb{Z}_p \hookrightarrow \widetilde{T}$. This gives rise to an embedding of relative Chow motives

$$\mathrm{br}\colon E \hookrightarrow (\iota, \nu)^\star \operatorname{Anc}_E(\widetilde{V})$$

with étale realisation $\mathbb{Q}_p \hookrightarrow (\iota, \nu)^\star \widetilde{\mathscr{V}}$, which we also denote by br. The following is an immediate application of the description of the $\bullet$-action in Lemma 6.6.

Corollary 6.8. *Let $\lambda = \sum_{i=1}^n c_i$. The morphism $\mathrm{br}\colon \mathbb{Q}_p \to (\iota, \nu)^\star \widetilde{\mathscr{V}}$ restricts to a morphism*

$$\mathbb{Z}_p[\lambda/n] \hookrightarrow (\iota, \nu)^\star \widetilde{\mathscr{T}}$$

of $\mathbf{H}(\mathbb{A}_f^p) \times \left(\widetilde{\Sigma} \cap \mathbf{H}(\mathbb{Q}_p)\right)$-equivariant sheaves.

6.4. **Functors from étale cohomology.** For any prime ℓ, we denote

$$G_\ell := \mathbf{G}(\mathbb{Q}_\ell), \quad G^\ell := \mathbf{G}(\mathbb{A}_f^\ell), \quad G_f := \mathbf{G}(\mathbb{A}_f)$$

and similarly for the groups $\mathbf{H}$ and $\mathbf{T}$. Let $\Sigma_{G_p} \subset G_p$ denote the open submonoid Σ in Lemma 6.6, and set $\Sigma_{H_p} = \Sigma_{G_p} \cap \mathbf{H}$. We let Υ_{G_p} be the collection of all compact open subgroups in Σ_{G_p} and take $\Upsilon_{G_p}^{\mathrm{tot}}$ to be the collection of all compact open subgroups in G_p.

Lemma 6.9. *There exists a (non-empty) maximal collection Υ_{G^p} of compact open subgroups of G^p, closed under inclusions and conjugation by G^p, such that the product $\Upsilon_{G^p} \times \Upsilon_{G_p}^{\mathrm{tot}}$ consists of sufficiently small subgroups of G_f. Moreover, for any $\ell \neq p$ and compact open subgroup $K_\ell \subset G_\ell$, there exists an element in Υ_{G^p} whose ℓ-adic component is equal to K_ℓ.*

Proof. Let K'_p denote any representative of the unique conjugacy class of maximal compact open subgroups in $G_p = \mathbb{Q}_p^\times \times \mathrm{GL}_{2n}(\mathbb{Q}_p)$. Then there exists a compact open subgroup K^p such that $K^p K'_p$ is sufficiently small and therefore, $K^p K_p$ is sufficiently small for any $K_p \subset G_p$. We take Υ_{G^p} to be the collection of all such groups K^p. This collection is closed under conjugation by G^p and contains all compact open subgroups of any element $K^p \in \Upsilon_{G^p}$. By deepening the level at a finite set of places S not containing ℓ and p, we can arrange that $K^p = K_\ell K_S K^{S\cup\{\ell,p\}} \in \Upsilon_{G^p}$ for any arbitrary K_ℓ. □

We now define the monoids $\Sigma_\square$ and collections $\Upsilon_\square$ that will be used throughout the paper. Firstly, we deal with the "tame level": set

- $\Sigma_{G^p}, \Sigma_{H^p}, \Sigma_{T^p}$ and $\Sigma_{\widetilde{G}^p}$ to be G^p, H^p, T^p and $\widetilde{G}^p$ respectively.
- Υ_{G^p} is the collection appearing in Lemma 6.9.
- Υ_{H^p} is defined similarly to Υ_{G^p}.
- $\Upsilon_{T^p} = \Upsilon_{T^p}^{\mathrm{tot}}$ denotes the collection of all compact open subgroups of T^p.
- $\Upsilon_{\widetilde{G}^p} = \Upsilon_{G^p} \times \Upsilon_{T^p}$ (we will only be interested in compact open subgroups of $\widetilde{G}^p$ of the form $K^p \times C^p$).

At the prime p we fix the following data:

- Σ_{G_p} and Σ_{H_p} are as in the start of this section. We set $\Sigma_{T_p} = T_p$ and
$$\Sigma_{\widetilde{G}_p} = \Sigma_{G_p} \times \Sigma_{T_p}$$
so $\Sigma_{\widetilde{G}_p}$ equals the monoid denoted $\widetilde{\Sigma}$ in Lemma 6.6.
- For $\square_p = G_p, H_p, T_p$ or $\widetilde{G}_p$, we set $\Upsilon_{\square_p}$ to be the collection of all compact open subgroups contained in $\Sigma_{\square_p}$ (so in particular, elements of $\Upsilon_{\widetilde{G}_p}$ are not necessarily of the form $K_p \times C_p$).

Finally, for $\square = G, H, T$ or $\widetilde{G}$, we set $\Sigma_\square = \Sigma_{\square^p} \times \Sigma_{\square_p}$ and $\Upsilon_\square = \Upsilon_{\square^p} \times \Upsilon_{\square_p}$. For brevity, we set
$$\mathcal{P}_\square := \mathcal{P}\left(\square_f, \Sigma_\square, \Upsilon_\square\right)$$
and similarly for $\mathcal{P}_{\square^p}$ and $\mathcal{P}_{\square_p}$.

Remark 6.10. Although not explicitly included in the notation, the collections $\Upsilon_{\square^p}$ depend on the group $\square_f$. The subscript simply refers to the ambient group of which the elements are compact open subgroups.

The following lemma will allow us to apply the results of section 5.

Lemma 6.11. *For a group $\square \in \{G, H, T, \widetilde{G}\}$ and a pair $(\Sigma, \Upsilon) \in \{(\Sigma_\square, \Upsilon_\square), (\Sigma_{\square_p}, \Upsilon_{\square_p}), (\Sigma_{\square^p}, \Upsilon_{\square^p})\}$ the axioms (T1)–(T3) at the start of section 5.1 are satisfied. Moreover, for any elements $U \in \Upsilon_H$ and $\widetilde{K} \in \Upsilon_G$, the intersection satisfies $U \cap \widetilde{K} \in \Upsilon_H$.*

Proof. By construction, if properties (T1)–(T3) are satisfied for both $(\Sigma_{\square_p}, \Upsilon_{\square_p})$ and $(\Sigma_{\square^p}, \Upsilon_{\square^p})$ then they are also satisfied for the pair $(\Sigma_\square, \Upsilon_\square)$. For the "tame level", the properties follow from the fact that the collection in Lemma 6.9 is closed under inclusions and conjugation. For the level at p, the properties follow from the fact that Υ is taken to be the collection of all compact open subgroups of Σ.

The last part of the lemma follows from the analogous statement at levels away and at p; the former is true because the collection in Lemma 6.9 is closed under inclusion, the latter is true because Υ_{H_p} is the collection of all compact open subgroups of Σ_{H_p}, which is also closed under inclusion. □

Let V be a self-dual algebraic representation of $\mathbf{G}_E$ with highest weight $\mathbf{c} = (0; c_1, \ldots, c_{2n})$ as in Definition 2.12. With notation as in section 6.3, we define the following cohomology functors on $\mathcal{P}_\square$ where $\square$ is to be replaced by the symbol of the group appearing in the subscript of the cohomology functor:

$$M_{H,\mathbb{Z}_p}(-) := \mathrm{H}^0_{\text{ét}}\left(\mathrm{Sh}_{\mathbf{H}}(-), \mathbb{Z}_p\right)[\lambda/n] \qquad M_{H,\mathbb{Q}_p}(-) := \mathrm{H}^0_{\text{ét}}\left(\mathrm{Sh}_{\mathbf{H}}(-), \mathbb{Q}_p\right)$$
$$M_{\widetilde{G},\mathbb{Z}_p}(-) := \mathrm{H}^{2n}_{\text{ét}}\left(\mathrm{Sh}_{\widetilde{\mathbf{G}}}(-), \widetilde{\mathscr{T}}(n)\right) \qquad M_{\widetilde{G},\mathbb{Q}_p}(-) := \mathrm{H}^{2n}_{\text{ét}}\left(\mathrm{Sh}_{\widetilde{\mathbf{G}}}(-), \widetilde{\mathscr{V}}(n)\right)$$
$$M_{G,\text{ét},\mathbb{Z}_p}(-) := \mathrm{H}^{2n-1}_{\text{ét}}\left(\mathrm{Sh}_{\mathbf{G}}(-)_{\bar{E}}, \mathscr{T}(n)\right) \qquad M_{G,\text{ét},\mathbb{Q}_p}(-) := \mathrm{H}^{2n-1}_{\text{ét}}\left(\mathrm{Sh}_{\mathbf{G}}(-)_{\bar{E}}, \mathscr{V}(n)\right)$$
$$M_{\widetilde{G},\text{ét},\mathbb{Z}_p}(-) := \mathrm{H}^{2n-1}_{\text{ét}}\left(\mathrm{Sh}_{\widetilde{\mathbf{G}}}(-)_{\bar{E}}, \widetilde{\mathscr{T}}(n)\right) \qquad M_{\widetilde{G},\text{ét},\mathbb{Q}_p}(-) := \mathrm{H}^{2n-1}_{\text{ét}}\left(\mathrm{Sh}_{\widetilde{\mathbf{G}}}(-)_{\bar{E}}, \widetilde{\mathscr{V}}(n)\right)$$

where $[\lambda/n]$ denotes the twist by the character $|\det|_p^{\lambda/n}$ as in section 6.3. If we do not wish to specify whether we are working integrally or rationally, we will simply leave the ring out of the subscript.

Remark 6.12. Note that the construction of the pullbacks and pushforwards in section 6.3 are valid in this setting because we have taken Υ_{G_p} to be the collection of all compact open subgroups in Σ_{G_p}; so we do indeed obtain cohomology functors.

Lemma 6.13. *The functors $M_H, M_{\widetilde{G}}, M_{G,\text{ét}}$ and $M_{\widetilde{G},\text{ét}}$ are CoMack functors on their respective categories. Moreover, we have a Cartesian pushforward $\iota_\star \colon M_H \to M_{\widetilde{G}}$.*

Proof. As explained in Example 5.2, all of these are cohomology functors satisfying axiom (Co). Note that $\mathbf{Z_H} = \mathrm{GU}(1) \times \mathrm{U}(1)$, $\mathbf{Z_G} = \mathrm{GU}(1)$, $\mathbf{Z_{\widetilde{G}}} = \mathrm{GU}(1) \times \mathrm{U}(1)$ and since $\mathrm{GU}(\mathbb{Z})$ and $\mathrm{U}(1)(\mathbb{Z})$ are finite, the $\mathbb{Z}$-points of all these central tori are finite too. As a result $\mathrm{GU}(1)(\mathbb{Q})$ (resp. $\mathrm{U}(1)(\mathbb{Q})$) is discrete in $\mathrm{GU}(1)(\mathbb{A}_f)$ (resp. $\mathrm{U}(1)(\mathbb{A}_f)$), so all the Shimura data satisfy axiom SV5 of [Mil03] (or (SD5) of Appendix B). This implies the action of any sufficiently small compact open subgroup (in the sense of Definition B.5) on the respective infinite-level Shimura variety is free, and the proper base change theorem applied to pullbacks of these Shimura varieties gives axiom (M).

For any compatible pair $(U, K) \in \Upsilon_H \times \Upsilon_{\widetilde{G}}$, we have a morphism of E-varieties $\mathrm{Sh}_{\mathbf{H}}(U) \to \mathrm{Sh}_{\widetilde{\mathbf{G}}}(K)$, and therefore we have a pushforward

$$(\iota_\star, \mathrm{br}) \colon (\mathrm{Sh}_{\mathbf{H}}(U), \mathbb{Z}_p[\lambda/n]) \to (\mathrm{Sh}_{\widetilde{\mathbf{G}}}(K), \widetilde{\mathscr{T}})$$

in the sense of Definition A.4.[4] By [Del71, Proposition 1.15], this forms an étale smooth E-pair of codimension n,[5] whence we also have a pushforward $\iota_\star \colon M_{H,\mathbb{Z}_p} \to M_{\widetilde{G},\mathbb{Z}_p}$ of cohomology functors by Proposition A.5, which is also Cartesian (in the sense of Definition 5.7) by the second point of Proposition A.5. Indeed, for any $K, L \in \Upsilon_{\widetilde{G}}$, $U \in \Upsilon_H$ with $U \subset K$, and any choice of representatives $\gamma \in U\backslash K/L$, the diagram given by

$$\begin{array}{ccc} \mathrm{Sh}_{\mathbf{H}}(U_\gamma) & \xrightarrow{\sqcup \gamma} & \mathrm{Sh}_{\widetilde{\mathbf{G}}}(L) \\ \downarrow & & \downarrow \\ \mathrm{Sh}_{\mathbf{H}}(U) & \xrightarrow{(\iota,\nu)} & \mathrm{Sh}_{\widetilde{\mathbf{G}}}(K) \end{array}$$

is Cartesian, where $U_\gamma = U \cap \gamma L \gamma^{-1}$ and $\gamma \colon \mathrm{Sh}_{\mathbf{H}}(U_\gamma) \to \mathrm{Sh}_{\widetilde{\mathbf{G}}}(L)$ is the morphism given by the composition $\mathrm{Sh}_{\mathbf{H}}(U_\gamma) \to \mathrm{Sh}_{\widetilde{\mathbf{G}}}(\gamma L \gamma^{-1}) \xrightarrow{\gamma} \mathrm{Sh}_{\widetilde{\mathbf{G}}}(L)$. ☐

Remark 6.14. Note that $(M_{H,\mathbb{Z}_p}, M_{H,\mathbb{Q}_p})$ and $(M_{\widetilde{G},\mathbb{Z}_p}, M_{\widetilde{G},\mathbb{Q}_p})$ do not form pairs of compatible cohomology functors as defined in the discussion preceding Corollary 5.12, because the natural maps are only $\mathbf{H}(\mathbb{A}_f^p) \times \mathbf{H}(\mathbb{Z}_p)$-equivariant and $\widetilde{\mathbf{G}}(\mathbb{A}_f^p) \times \widetilde{J}$-equivariant respectively.

6.5. **Local finiteness of the Hecke action.** In the next subsection, we will need certain finiteness hypothesis on the classes in the image of $\iota_\star$. We recall the following notion from [Pil18].

Definition 6.15. Let M be a $\mathbb{Z}_p$-module and A an endomorphism of M. We say that the action of A on M is *locally finite* if for all $n \in \mathbb{N}$ and all $v \in M$, the elements $\{A^k v\}_{k \in \mathbb{N}}$ generate a finite $\mathbb{Z}/p^n\mathbb{Z}$-submodule of $M/p^n M$.

Lemma 6.16. *Let M be a $\mathbb{Z}_p$-module and A an endomorphism of M. There exists a unique maximal sub-module $M^{\mathrm{lf}} \subset M$ such that the action of A on M^{lf} is locally finite. Furthermore, there exists an endomorphism e_{ord} of M^{lf} such that A commutes with e_{ord} and gives a decomposition*

$$M^{\mathrm{lf}} = e_{\mathrm{ord}} \cdot M^{\mathrm{lf}} \oplus (\mathrm{id} - e_{\mathrm{ord}}) M^{\mathrm{lf}}$$

with A acting invertibly on the first factor. Moreover, this construction is functorial in the pair (M, A), i.e. if (N, B) is another such pair and $\varphi \colon M \to N$ is a homomorphism of $\mathbb{Z}_p$-modules that intertwines the actions of A and B, then there is an induced homomorphism $\varphi^{\mathrm{lf}} \colon M^{\mathrm{lf}} \to N^{\mathrm{lf}}$ which respects the decomposition above.

[4] To ease notation we will simply denote this by $\iota_\star$ rather than the more cumbersome notation $(\iota, \nu)_\star$.

[5] More precisely, [Del71, Proposition 1.15] implies that there exists a compact open subgroup L containing U such that $\mathrm{Sh}_{\mathbf{H}}(U) \to \mathrm{Sh}_{\widetilde{\mathbf{G}}}(L)$ is a closed immersion. This map factors through the morphism $\mathrm{Sh}_{\mathbf{H}}(U) \to \mathrm{Sh}_{\widetilde{\mathbf{G}}}(L \cap K)$, which by the cancellation property, is also a closed immersion. The desired factorisation is then given by $\mathrm{Sh}_{\mathbf{H}}(U) \to \mathrm{Sh}_{\widetilde{\mathbf{G}}}(L \cap K) \to \mathrm{Sh}_{\widetilde{\mathbf{G}}}(K)$.

Proof. We define M^{lf} to be subset of all $x \in M$ such that for all n, the elements $\{A^k \bar{x}_n\}_k$ generate a finite sub-module of M/p^n, where $\bar{x}_n$ is the reduction of x modulo p^n. Clearly this is a A-stable sub-module of M. If $x \in M^{\mathrm{lf}}$ and $y = \varphi(x)$, then $\{B^k \bar{y}_n\}_k = \varphi(\{A^k \bar{x}_n\}_k)$ which implies that $y \in N^{\mathrm{lf}}$. The existence of e_{ord} and the corresponding decomposition is [Pil18, Lemma 2.1.3], and its funtoriality is straightforward. □

The following lemma shows that the image of the pushforward $\iota_\star$ lands in the locally-finite subspace of $M_{\widetilde{G}}$.

Lemma 6.17. *Let $(U, K \times C) \in \Upsilon_H \times \Upsilon_{\widetilde{G}}$ be a compatible pair. Then the image of*

$$\iota_\star \colon M_{H,\mathbb{Z}_p}(U) \to M_{\widetilde{G},\mathbb{Z}_p}(K \times C)$$

is contained in $M_{\widetilde{G},\mathbb{Z}_p}(K \times C)^{\mathrm{lf}}$, the locally finite sub-module with respect to the endomorphism $\mathcal{T}$.

Proof. To ease notation, we denote

$$X := \mathrm{Sh}_{\mathbf{H}}(U), \qquad Y := \mathrm{Sh}_{\widetilde{\mathbf{G}}}(K \times C) = \mathrm{Sh}_{\mathbf{G}}(K) \times_{\mathrm{Spec}\, E} \mathrm{Sh}_{\mathbf{T}}(C)$$

and $f \colon X \to Y$ the finite unramified morphism induced from the map on Shimura datum. Let K_p denote the projection of K to $G(\mathbb{Q}_p)$ and fix a nested collection of normal compact open subgroups

$$K_{p,i} \subset K_p \qquad (i \geq 1)$$

that has trivial intersection. Set $K_{p,0} = K_p$ and let K_i denote set of $\gamma \in K$ such that $\gamma_p \in K_{p,i}$. If we set $Y_i = \mathrm{Sh}_{\widetilde{G}}(K_i \times C)$, then there exists a finite set of primes S of E and a pro-system of smooth quasi-projective finite-type schemes $(\mathcal{Y}_i)_{i \geq 0}$ over $\mathcal{O}_{E,S}$ such that:

- For each i, there exist isomorphisms
$$p_i \colon \mathcal{Y}_{i,E} \xrightarrow{\sim} Y_i$$
which are compatible under varying i.
- For each $K_{p,j} \subset g K_{p,i} g^{-1}$ with $g \in \widetilde{G}_p$, the morphism $Y_j \xrightarrow{\cdot g} Y_i$ extends to a morphism
$$\mathcal{Y}_j \xrightarrow{\cdot g} \mathcal{Y}_i$$
with respect to the isomorphisms above.
- For $j \geq i$, the map
$$\mathcal{Y}_j \to \mathcal{Y}_i$$
is étale with Galois group $K_{p,i}/K_{p,j}$.

Indeed, since $(\mathbf{G}, X_{\mathbf{G}})$ is a PEL-type Shimura variety, the existence of integral models for $\mathrm{Sh}_{\mathbf{G}}(K_i)$ is guaranteed by [Lan13, Theorem 1.4.1.12] once we fix a choice of PEL datum, and we can replace $\mathrm{Sh}_{\mathbf{T}}(C)$ with the union of the spectra of the ring of integers of a collection of finite extensions of E localised at all the places above S (in fact, it will turn out that we only need one, as $\mathrm{Sh}_{\mathbf{T}}$ will be a torsor over the anticylotomic extension). Similarly, there exists an integral model for X and a pro-system of schemes $(\mathcal{X}_i)_{i \geq 0}$ for $(\mathbf{H}, X_{\mathbf{H}})$ satisfying the analogous conditions as above (see Example B.2).

Set $\mathcal{Y} := \mathcal{Y}_0$ and $\mathcal{X} := \mathcal{X}_0$. The quadruplet $(\mathcal{Y}, \{\mathcal{Y}_i\}, K_p, \widetilde{T})$ determines a lisse $\mathbb{Z}_p$-sheaf $\widetilde{\mathfrak{T}}$ on $\mathcal{Y}_K$ (e.g. see [HT01, §III.2]) and it is easily verified that the pullback of $\widetilde{\mathfrak{T}}$ under $p \colon Y \to \mathcal{Y}$ is isomorphic to $\widetilde{\mathscr{T}}$. A similar construction holds for the quadruplets $(\mathcal{X}, (\mathcal{X}_i), U_p, \widetilde{T})$ and $(\mathcal{X}, (\mathcal{X}_i), U_p, \mathbb{Z}_p[\lambda/n])$ with respect to the map $q \colon X \to \mathcal{X}$, where $\mathbb{Z}_p[\lambda/n]$ is the equivariant sheaf defined in section 6.3. We note that since the branching law holds on X, it also holds on $\mathcal{X}$.

By Theorem 3.2.1 in [Poo17], we may choose (after possibly enlarging S) an extension of f to a morphism $g \colon \mathcal{X} \to \mathcal{Y}$ which makes $(\mathcal{X}, \mathcal{Y})$ into an étale smooth $\mathcal{O}_{E,S}$-pair of codimension n, as in Definition A.4. We may also assume that the sheaf on $\mathcal{X}$ is identified with $f^\star \widetilde{\mathfrak{T}}$ and that $\pi_0(\mathcal{X})$ is identified with $\pi_0(X)$. This data therefore determines a pushforward

$$(g, \mathrm{br})_\star \colon (\mathcal{X}, \mathbb{Z}_p[\lambda/n]) \to (\mathcal{Y}, \widetilde{\mathfrak{T}}).$$

Since the diagram on the left below is Cartesian, the diagram on the right is commutative by Proposition A.5.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ \mathcal{X} & \xrightarrow{g} & \mathcal{Y} \end{array} \qquad \begin{array}{ccc} M_{H,\mathbb{Z}_p}(U) & \xrightarrow{\iota_\star} & M_{\widetilde{G},\mathbb{Z}_p}(K \times C) \\ \| & & \uparrow{\scriptstyle q^\star} \\ \mathrm{H}^0_{\text{ét}}(\mathcal{X}, \mathbb{Z}_p)[\lambda/n] & \xrightarrow{(g,\mathrm{br})_\star} & \mathrm{H}^{2n}_{\text{ét}}(\mathcal{Y}, \widetilde{\mathfrak{T}}(n)) \end{array}$$

Our choices above imply that $q^\star$ is a $\mathbb{Z}_p[\mathcal{T}]$-module homomorphism.

Finally, to conclude the proof of the lemma, we will show that $\mathrm{H}^{2n}_{\text{ét}}(\mathcal{Y}, \widetilde{\mathfrak{T}}(n))$ is a finitely-generated $\mathbb{Z}_p$-module. Enlarge S to include all primes above p, if necessary. By [Mil06, Proposition II.2.9], the étale cohomology of $\operatorname{Spec} \mathcal{O}_{E,S}$ for constructible étale sheaves is identified with the group cohomology of $\mathrm{Gal}(E_S/E)$ (where E_S is the maximal unramified outside S extension of E). Therefore, for any integer $r \geq 1$ and any constructible étale sheaf $\mathscr{F}$ of $\mathbb{Z}/p^r\mathbb{Z}$-modules, the cohomology group $\mathrm{H}^i_{\text{ét}}(\operatorname{Spec} \mathcal{O}_{E,S}, \mathscr{F})$ is a finite $\mathbb{Z}/p^r\mathbb{Z}$-module. Indeed, this is because $\mathrm{Gal}(E_S/E)$ satisfies condition (F) (see Lemma A.11).

By [Kat93, §3.1.3], the higher direct images of $\widetilde{\mathfrak{T}}(n)/p^r$ under the morphism $\mathcal{Y} \xrightarrow{t} \operatorname{Spec} \mathcal{O}_{E,S}$ are constructible. Combining this with the Grothendieck spectral sequence associated with the composition

$$R\Gamma_{\text{ét}}(\mathcal{Y}, -) = R\Gamma_{\text{ét}}(\operatorname{Spec} \mathcal{O}_{E,S}, -) \circ Rt_*$$

(where $R\Gamma_{\text{ét}}(Z, -)$ denotes the right-derived global sections of constructible sheaves on Z), we see that $\mathrm{H}^{2n}_{\text{ét}}\left(\mathcal{Y}, \widetilde{\mathfrak{T}}(n)/p^r\right)$ is a finite $\mathbb{Z}/p^r\mathbb{Z}$-module, for $r \geq 1$. This system of finite-type modules therefore satisfies the Mittag-Leffler property ([Jan88, Remark 1.14]), and we have

$$\mathrm{H}^{2n}_{\text{ét}}\left(\mathcal{Y}, \widetilde{\mathfrak{T}}(n)\right) = \varprojlim_r \mathrm{H}^{2n}_{\text{ét}}\left(\mathcal{Y}, \widetilde{\mathfrak{T}}(n)/p^r\right).$$

The right-hand side is an inverse limit of finite-type modules, so is finite-type over $\mathbb{Z}_p$ (see the proof of Proposition 4.2.3 in [Nek06], for example). □

6.6. **Ordinary and finite-slope parts.** To obtain classes in the cohomology of $\mathrm{Sh}_{\mathbf{G}}$ that are cohomologically trivial, it turns out it is enough to construct universal norms out of them in the "p-direction". Once we have accomplished this, we can apply the construction in Proposition A.10 to map to the cohomology of the Galois representation attached to π. We obtain these universal norms by projecting these classes to the ordinary part of the cohomology of $\mathrm{Sh}_{\widetilde{\mathbf{G}}}$ under the action of a suitable idempotent. As a consequence, we also obtain the vertical norm relations for the Euler system.

We fix the level in G_p component to be $J \in \Upsilon_{G_p}$ in this section (although results apply for any fixed level K_p). We note that each $M_{\widetilde{G},\mathbb{Z}_p}(K^pJ \times C)$ is endowed with an action of $\mathcal{T}$, and these actions commute with pullbacks and pushforwards. More precisely

Lemma 6.18. *Let $M^p_{\widetilde{G},\mathbb{Z}_p} : \mathcal{P}_{\widetilde{G}^p} \times \mathcal{P}_{T_p} \to \mathbb{Z}_p$-**Mod** denote the functor*

$$(K^p \times C^p, C_p) \mapsto M_{\widetilde{G},\mathbb{Z}_p}(K^pJ \times C^pC_p).$$

*Then $\mathcal{T}$ is an endo-functor of $M^p_{\widetilde{G},\mathbb{Z}_p}$. In other words, the functor $M^p_{\widetilde{G},\mathbb{Z}_p}$ is valued in $\mathbb{Z}_p[\mathcal{T}]$-**Mod**.*

Proof. This is a consequence of Lemma 5.6. □

The ordinary idempotent is now constructed on the sub-functor on which $\mathcal{T}$ acts locally finitely.

Lemma 6.19. *The mapping $M^{\mathrm{lf}}_{\widetilde{G},\mathbb{Z}_p} : \mathcal{P}_{\widetilde{G}^p} \times \mathcal{P}_{T_p} \to \mathbb{Z}_p$-**Mod** given by*

$$(K^p \times C^p, C_p) \mapsto M_{\widetilde{G},\mathbb{Z}_p}(K^pJ \times C^pC_p)^{\mathrm{lf}}$$

is a $\mathbb{Z}_p[\mathcal{T}]$-valued CoMack sub-functor of $M^p_{\widetilde{G},\mathbb{Z}_p}$. Moreover, there exists an idempotent endofunctor e_{ord} of $M^{\mathrm{lf}}_{\widetilde{G},\mathbb{Z}_p}$ giving a decomposition

$$M^{\mathrm{lf}}_{\widetilde{G},\mathbb{Z}_p} = e_{\mathrm{ord}} \cdot M^{\mathrm{lf}}_{\widetilde{G},\mathbb{Z}_p} \oplus (\mathrm{id} - e_{\mathrm{ord}}) \cdot M^{\mathrm{lf}}_{\widetilde{G},\mathbb{Z}_p}$$

which commutes with $\mathcal{T}$, and the action of $\mathcal{T}$ on the first component is invertible.

Proof. Axioms (C1), (C2), (C3) and (M) hold by the first part of Lemma 6.16 and Lemma 5.6. Axiom (Co) holds by the additivity of taking locally finite parts. The rest follows from the second part Lemma 6.16. □

Definition 6.20. We consider the following cohomology functors

$$M^{\mathrm{ord}}_{\widetilde{G},\mathbb{Z}_p}\colon \mathcal{P}_{\widetilde{G}^p}\times\mathcal{P}_{T_p}\to\mathbb{Z}_p[\mathcal{T},\mathcal{T}^{-1}]\text{-}\mathbf{Mod} \qquad M^{\mathrm{fs}}_{\widetilde{G},\mathbb{Z}_p}\colon \mathcal{P}_{\widetilde{G}^p}\times\mathcal{P}_{T_p}\to\mathbb{Z}_p[\mathcal{T},\mathcal{T}^{-1}]\text{-}\mathbf{Mod}$$

$$(K^p\times C^p,C_p)\mapsto e_{\mathrm{ord}}\cdot M^{\mathrm{lf}}_{G,\mathbb{Z}_p}(K^pJ\times C^pC_p) \qquad (K^p\times C^p,C_p)\mapsto\Big(M^{\mathrm{lf}}_{G,\mathbb{Z}_p}(K^pJ\times C^pC_p)\Big)_{\mathcal{T}}$$

which we refer to as *ordinary part* and the *finite slope part* of $M_{\widetilde{G},\mathbb{Z}_p}$ respectively. Here the subscript denotes localisation. We denote the natural transformation $M^{\mathrm{fs}}_{\widetilde{G},\mathbb{Z}_p}\to M^{\mathrm{ord}}_{\widetilde{G},\mathbb{Z}_p}$ by e_{ord} as well.

Lemma 6.21. *The map $M^{\mathrm{ord}}_{\widetilde{G},\mathrm{Iw},\mathbb{Z}_p}\colon\mathcal{P}_{\widetilde{G}^p}\to\mathbb{Z}_p[\mathcal{T},\mathcal{T}^{-1}]$-**Mod** given by*

$$K^p\times C^p\mapsto\varprojlim_{C_p}M^{\mathrm{ord}}_{\widetilde{G},\mathbb{Z}_p}(K^pJ\times C^pC_p)$$

defines a CoMack functor.

Proof. All the axioms hold for $M^{\mathrm{ord}}_{\widetilde{G},\mathbb{Z}_p}$ because they hold for $M^{\mathrm{lf}}_{\widetilde{G},\mathbb{Z}_p}$ and e_{ord} is additive. The pushforwards along $\mathcal{P}_{T^p}$ commute with pushforwards/pullbacks along $\mathcal{P}_{\widetilde{G}^p}$ (because the operators $\mathcal{T}^N$ also do) by Lemma 5.6, and therefore they also hold for $M^{\mathrm{fs}}_{\widetilde{G},\mathrm{Iw},\mathbb{Z}_p}$ by functoriality of the inverse limit. □

Let $M^p_{H,\mathbb{Z}_p}\colon\mathcal{P}_{H^p}\to\mathbb{Z}_p$-**Mod** denote the functor $U^p\mapsto M_{H,\mathbb{Z}_p}(U^pH(\mathbb{Z}_p))$.

Proposition 6.22. *There is a (Cartesian) pushforward of cohomology functors*

$$\iota^{\mathrm{ord}}_\star\colon M^p_{H,\mathbb{Z}_p}\to M^{\mathrm{ord}}_{\widetilde{G},\mathrm{Iw},\mathbb{Z}_p}.$$

Proof. We apply the results of [Loe19] to the pushforward $\iota_\star\colon M_{H,\mathbb{Z}_p}\to M_{\widetilde{G},\mathbb{Z}_p}$. Let $V_r\in\mathcal{P}_{\widetilde{G}_p}$ and $D_{p^r}\in\mathcal{P}_{T_p}$ be as in section 6.2. By Lemma 6.2 and Theorem 4.5.3 in *op.cit.*, we have a (Cartesian) pushforward

$$M^p_{H,\mathbb{Z}_p}\to\varprojlim_r M_{\widetilde{G},\mathbb{Z}_p}(-\cdot V_r)_{\mathcal{T}} \tag{6.23}$$

where the target is a functor on $\mathcal{P}_{\widetilde{G}^p}$. Explicitly, let $\eta_r=\widetilde{u}\widetilde{\tau}^r$ and $U^p=H^p\cap K^p$. For all $r\geq 1$, we have a map θ_{r,K^p} given by the composition:

$$\begin{aligned} M_{H,\mathbb{Z}_p}(U^p\cdot H(\mathbb{Z}_p)) &\xrightarrow{\mathrm{pr}^\star} M_{H,\mathbb{Z}_p}(U^p\cdot(H_p\cap\eta_rV_r\eta_r^{-1}))\\ &\xrightarrow{\iota_\star} M_{\widetilde{G},\mathbb{Z}_p}(K^p\cdot(\eta_rV_r\eta_r^{-1}))\\ &\xrightarrow{[\eta_r]_\star} M_{\widetilde{G},\mathbb{Z}_p}(K^p\cdot V_r)\\ &\xrightarrow{\mathcal{T}^{-r}} M_{\widetilde{G},\mathbb{Z}_p}(K^p\cdot V_r)_{\mathcal{T}}\end{aligned}$$

For any r, the maps θ_{r,K_p}, θ_{r+1,K^p} commute with the pushforward map from level $K^p\cdot V_{r+1}$ to $K^p\cdot V_r$ – this amounts to showing the commutativity of the bottom left square of the diagram in §4.5 of *op. cit*, which is where the Cartesian property of $\iota_\star$ is invoked. Moreover, this pushforward is also compatible with the $\mathcal{T}$ action – this follows from Lemma 6.4 (a) and the Cartesian property of $M_{\widetilde{G},\mathbb{Z}_p}$. The upshot is that we have an inverse system of maps $(\theta_{r,K_p})_{r\geq1}$ of $\mathbb{Z}_p[\mathcal{T},\mathcal{T}^{-1}]$-modules. As the dependence on K^p is functorial by Lemma 5.6, this gives the pushforward claimed in (6.23).

Now $V_r\subset J\times D_{p^r}$, so there also exists projection maps

$$\mathrm{pr}_\star\colon M_{\widetilde{G}}(-\cdot V_r)\longrightarrow M_{\widetilde{G}}(-\cdot(J\times D_{p^r})).$$

Again, by Lemma 5.6, this is a pushforward of cohomology functors, and moreover commutes with the action of $\mathcal{T}$ by Lemma 6.4 (b), whence also with the localisation with respect to $\mathcal{T}$. We therefore obtain a pushforward of cohomology functors

$$\iota^{\mathrm{fs}}_\star\colon M^p_{H,\mathbb{Z}_p}\to\varprojlim_r M_{\widetilde{G},\mathbb{Z}_p}(-\cdot(J\times D_{p^r}))_{\mathcal{T}}$$

by composing (6.23) with $\mathrm{pr}_\star$ (after localizing at $\mathcal{T}$). Finally, we claim that this functor factors through $\varprojlim_r M^{\mathrm{fs}}_{\widetilde{G},\mathbb{Z}_p}(-\cdot(J\times D_{p^r}))$. Indeed, it suffices to show that the image of the map

$$M^p_{H,\mathbb{Z}_p}(U^p\cdot H(\mathbb{Z}_p))\to M_{\widetilde{G},\mathbb{Z}_p}(K^pJ\times C^pD_{p^r})$$

lands in $M^{\mathrm{lf}}_{\widetilde{G},\mathbb{Z}_p}(K^pJ \times C^pD_{p^r})$ which was proven in Lemma 6.17. We then set

$$\iota^{\mathrm{ord}}_\star \colon M^p_{H,\mathbb{Z}_p} \to M^{\mathrm{ord}}_{\widetilde{G},\mathrm{Iw},\mathbb{Z}_p}$$

to be $e_{\mathrm{ord}} \circ \iota^{\mathrm{fs}}_\star$, which completes the proof. □

6.7. **Pushforward to Iwasawa cohomology.** For a compact open subgroup $C \in \Upsilon_T$, we consider the finite group

$$\Delta(C) := \mathbf{T}(\mathbb{Q})\backslash\mathbf{T}(\mathbb{A}_f)/C$$

which carries an action of $\mathrm{Gal}(E^{\mathrm{ab}}/E)$ via translation by the character $\kappa_C = \kappa$, where we set $\kappa_C(\sigma) = \bar{s}_f/s_f$ for any idele $s \in \mathbb{A}_E^\times$ satisfying $\mathrm{Art}_E(s) = \sigma$. We let $E(C)$ denote the finite abelian extension of E corresponding to the kernel of this character. Since $M_{\widetilde{G},\acute{e}\mathrm{t},\mathbb{Z}_p}$ and $M_{G,\acute{e}\mathrm{t},\mathbb{Z}_p}$ are finitely-generated over $\mathbb{Z}_p$ (see [Del77, Corollaire 1.10, Chapitre 7]), the action of $\mathcal{T}$ on both of these modules is locally finite, and we can therefore define ordinary and finite-slope parts on the whole functor.

We note the following application of Shapiro's lemma.

Lemma 6.24. *We have an isomorphism*

$$\mathrm{H}^1\left(E, M_{\widetilde{G},\acute{e}\mathrm{t},\mathbb{Z}_p}(K \times C)\right) \cong \mathrm{H}^1\left(E(C), M_{G,\acute{e}\mathrm{t},\mathbb{Z}_p}(K)\right)$$

which intertwines the action of $\mathrm{Gal}(E(C)/E)$ *and* $\mathbf{T}(\mathbb{A}_f)$ *via the character* κ_C^{-1} *(explicitly, the action of an element* $\sigma \in \mathrm{Gal}(E(C)/E)$ *on the right-hand side corresponds to the action of* $\kappa_C(\sigma)^{-1}$ *on the left-hand side). The same statement holds for ordinary and finite-slope parts.*

Proof. One sees from the definition of the Shimura datum for $\mathbf{T}$ that $\mathrm{Gal}(E^{\mathrm{ab}}/E)$ acts on $\mathrm{Sh}_{\mathbf{T}}(C)_{\bar{E}} = \Delta(C)$ via translation by the character $\kappa = \kappa_C$. One has an isomorphism

$$\begin{aligned}(6.25)\quad \mathrm{H}^0_{\acute{e}\mathrm{t}}\left(\mathrm{Sh}_{\mathbf{T}}(C) \times_{\mathrm{Spec}\, E} \mathrm{Spec}\, \overline{E}, \mathbb{Z}_p\right) = \mathrm{Maps}\left(\mathrm{Sh}_{\mathbf{T}}(C)(\mathbb{C}), \mathbb{Z}_p\right) &= \mathbb{Z}_p[\mathrm{Gal}(E(C)/E)] \\ \phi &\mapsto \sum_{\sigma \in \mathrm{Gal}(E(C)/E)} \phi(\kappa(\sigma^{-1})) \cdot \sigma\end{aligned}$$

which intertwines the action of $t \in \mathbf{T}(\mathbb{A}_f)$ on the left-hand side (given by right-translation of the argument $(t \cdot \phi)(-) = \phi(- \cdot t)$) with the action of $\sigma \in \mathrm{Gal}(E(C)/E)$ on the right-hand side (the action is by left-multiplication), where σ is the unique element such that the image of t in $\Delta(C)$ equals $\kappa(\sigma)$.

We have the following isomorphisms

$$\begin{aligned}(6.26)\quad \mathrm{H}^1\left(E, M_{\widetilde{G},\acute{e}\mathrm{t},\mathbb{Z}_p}(K \times C)\right) &\cong \mathrm{H}^1\left(E, M_{G,\acute{e}\mathrm{t},\mathbb{Z}_p}(K) \otimes_{\mathbb{Z}_p} \mathrm{H}^0_{\acute{e}\mathrm{t}}\left(\mathrm{Sh}_{\mathbf{T}}(C) \times_{\mathrm{Spec}\, E} \mathrm{Spec}\, \overline{E}, \mathbb{Z}_p\right)\right) \\ (6.27)\quad &\cong \mathrm{H}^1\left(E, M_{G,\acute{e}\mathrm{t},\mathbb{Z}_p}(K) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\mathrm{Gal}(E(C)/E)]\right) \\ (6.28)\quad &\cong \mathrm{H}^1\left(E(C), M_{G,\acute{e}\mathrm{t},\mathbb{Z}_p}(K)\right)\end{aligned}$$

where (6.26), (6.27) and (6.28) follow from the Künneth formula, (6.25) and Shapiro's lemma respectively. The action of an element $\sigma \in \mathrm{Gal}(E(C)/E)$ on the right-hand side of (6.28) is intertwined with left multiplication by σ^{-1} on $\mathbb{Z}_p[\mathrm{Gal}(E(C)/E)]$ appearing on the right-hand side of (6.27) (see [Nek06, §8.2]). By (6.25), this is precisely intertwined with the action of an element $t^{-1} \in \mathbf{T}(\mathbb{A}_f)$ on the left-hand side of (6.26), where the image of t in $\Delta(C)$ coincides with $\kappa(\sigma)$.

The same result holds for the finite-slope and ordinary parts because these operations commute with the action of $\mathbf{T}$. □

Definition 6.29. We let $N_{\mathrm{Iw},\mathbb{Z}_p} \colon \mathcal{P}_{\widetilde{G}^p} \to \mathbb{Z}_p[\mathcal{T}.\mathcal{T}^{-1}]$**-Mod** denote the map

$$K^p \times C^p \mapsto \varprojlim_r \mathrm{H}^1\left(E, M^{\mathrm{ord}}_{\widetilde{G},\acute{e}\mathrm{t},\mathbb{Z}_p}(K^pJ \times C^pD_{p^r})\right).$$

Lemma 6.30. $N_{\mathrm{Iw},\mathbb{Z}_p}$ *is a CoMack functor.*

Proof. The functor $N_{\mathrm{Iw},\mathbb{Z}_p}$ is obtained by composing the functor $M_{\widetilde{G},\acute{e}\mathrm{t},\mathbb{Z}_p}(K^pJ \times C^pD_{p^r})^{\mathrm{fs}}$ with e_{ord}, applying $\mathrm{H}^1(E,-)$ and then taking the inverse limit over r. All of these operations are additive endomorphisms of the category of $\mathbb{Z}_p[\mathcal{T},\mathcal{T}^{-1}]$-modules. □

We now obtain the vertical norm relations for our Euler system. Set $M_{H,\mathbb{Q}_p} := M^p_{H,\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $N_{\mathrm{Iw},\mathbb{Q}_p} := N_{\mathrm{Iw},\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Theorem 6.31 (Vertical norm relations)**.** *Set $E_r(C^p) = E(C^p D_{p^r})$ and $E_\infty(C^p) = \bigcup E_r(C^p)$. For $R = \mathbb{Z}_p$ or $\mathbb{Q}_p$, there is a (Cartesian) pushforward of cohomology functors*

$$\iota_{\star,\mathrm{Iw}} \colon M^p_{H,R} \to N_{\mathrm{Iw},R}$$

which is compatible with respect to base change of coefficients (as in the paragraph preceding Corollary 5.12).

Proof. We first note that by Lemma 6.24 that the functor $N_{\mathrm{Iw},\mathbb{Z}_p}$ is isomorphic to the functor

$$K^p \times C^p \mapsto \varprojlim_r \mathrm{H}^1\left(E_r(C^p), M^{\mathrm{ord}}_{G,\text{ét},\mathbb{Z}_p}(K^p J)\right)$$

where the inverse limit is with respect to corestriction. Moreover, the fields $E_r(C^p)$ for $r \geq 0$ form Γ-towers for $\Gamma = \mathbb{Z}_p^\times$ in the sense of Appendix A.2. By Proposition A.10 applied to the Shimura varieties of level K^p and Γ-towers of level C^p, we get an Abel–Jacobi map:

$$\mathrm{AJ}_{\mathrm{Iw}} \colon \varprojlim_r M_{\widetilde{G},\mathbb{Z}_p}(K^p J \times C^p D_{p^r}) = M^{2n}_{\mathrm{Iw}}\left(E_\infty(C^p), \mathrm{Sh}_{\mathbf{G}}(K^p J)\right)$$
$$\to \mathrm{H}^1_{\mathrm{Iw}}\left(E_\infty(C^p), \overline{M}^{2n-1}(\mathrm{Sh}_{\mathbf{G}}(K^p J))\right) = \varprojlim_r \mathrm{H}^1\left(E_r(C^p), M_{G,\text{ét},\mathbb{Z}_p}(K^p J)\right)$$

because the p-adic Lie extension $E_\infty(C^p)$ has positive dimension. By the functoriality properties in *loc.cit.*, this map commutes with taking ordinary parts and assembles into a map of cohomology functors:

$$\mathrm{AJ}_{\mathrm{Iw}} \colon M^{\mathrm{ord}}_{\widetilde{G},\mathrm{Iw},\mathbb{Z}_p} \to \varprojlim_r \mathrm{H}^1\left(E_r(-), M^{\mathrm{ord}}_{G,\text{ét},\mathbb{Z}_p}(- \cdot J)\right) = N_{\mathrm{Iw},\mathbb{Z}_p}.$$

The map $\iota_{\star,\mathrm{Iw}}$ is then defined as the composition $\iota_{\star,\mathrm{Iw}} := \mathrm{AJ}_{\mathrm{Iw}} \circ \iota^{\mathrm{ord}}_\star$. The result for $\mathbb{Q}_p$ follows by inverting p. □

7. The anticyclotomic Euler system

We now have all the ingredients needed to define the Euler system. Let π_0 be a cuspidal automorphic representation of $\mathbf{G}_0$ such that $\pi_{0,\infty}$ lies in the discrete series, and let π be a lift to $\mathbf{G}$. From now on, we make the following assumption:

Assumption 7.1. *We assume that:*

(1) $\pi_{0,f}$ admits a $\mathbf{H}_0(\mathbb{A}_f)$-linear model. In particular, we can choose a lift π with trivial central character (see section 2.4).

(2) n is odd and π satisfies Assumption 3.5. In particular, let K be a compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$ such that $\pi_f^K \neq 0$.

(3) Let S be the finite set of primes containing all primes that ramify in E and all primes where K is not hyperspecial. We assume that $p \notin S$ and splits in $E/\mathbb{Q}$. In this case we set $S' = S \cup \{p\}$.

(4) We have

$$K = K_p \cdot K_S \cdot K^{S'}$$

with K_S satisfying the following property: for any compact open subgroups $L_p \subset G_p$ and $L^{S'} \subset G^{S'}$, the subgroup $L_p K_S L^{S'}$ is sufficiently small (see section 2.2).

(5) Let $\mathbf{c} = (0; c_1, \ldots, c_{2n})$ denote the weight corresponding to the discrete series L-packet containing π_∞, and set $\lambda = \sum_{i=1}^n c_i$. We assume that π is "Siegel ordinary", i.e. there exists an eigenvector

$$\varphi = \left(\bigotimes_{\ell \notin S'} \varphi_\ell\right) \otimes \varphi_S \otimes \varphi_p \in \bigotimes_{\ell \notin S'} \pi_\ell^{K_\ell} \otimes \pi_S^{K_S} \otimes \pi_p^J = \pi_f^{K^p J}$$

of the Hecke operator $\mathcal{U}_S := p^\lambda [J\tau J]$ with eigenvalue α a p-adic unit. Here, J is the Siegel parahoric subgroup and τ is the element defined in section 6.1. In particular, this implies that α occurs as an eigenvalue for the transpose operator $\mathcal{U}'_S := p^\lambda [J\tau^{-1} J]$ on $(\pi_f^\vee)^{K^p J}$.

Remark 7.2 (Justification of Assumption 7.1, (5)). Suppose that $\pi_{0,p}$ is *generic*, in the sense that $\pi_{0,p} \cong I(\chi)$, where $I(-)$ denotes the (normalised) induction from the (standard) Borel to $\mathbf{G}_0(\mathbb{Q}_p) \cong \mathrm{GL}_{2n}(\mathbb{Q}_p)$, and $\chi = \chi_1 \boxtimes \cdots \boxtimes \chi_{2n} \colon \mathbb{Q}_p^{2n} \to \mathbb{C}^\times$ is an unramified character, naturally viewed as a character of the (standard) torus. Let $\beta_i = \chi_i(p)$ $(i = 1, \ldots, 2n)$ denote the Satake parameters of $\pi_{0,p}$. For any rational number s, we let p^s denote the unique element of $\overline{\mathbb{Q}}$ which maps to $e^{s \log p}$ under the fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Via the fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, we can also view p^s as an element of $\overline{\mathbb{Q}}_p$.

Let $v|p$ be the prime of E fixed by the embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, and set $\rho_v := \rho_\pi|_{E_v}$. Then by Theorem 3.4(3), the Galois representation ρ_v is crystalline and one has an equality of polynomials

$$\det\left(1 - \varphi \cdot X | \mathbf{D}_{\mathrm{cris}}(\rho_v)\right) = \prod_{i=1}^{2n} (1 - p^{n-1/2}\beta_i \cdot X).$$

Let v_p denote the p-adic valuation on $\overline{\mathbb{Q}}_p$, normalised so that $v_p(p) = 1$, and order the Satake parameters so that $\gamma_i := p^{n-1/2}\beta_i$ satisfy $v_p(\gamma_1) \leq \cdots \leq v_p(\gamma_{2n})$. For any subset $I \subset \{1, \ldots, 2n\}$ of size n, set $\gamma_I = \prod_{i \in I} \gamma_i$.

Recall that attached to $D = \mathbf{D}_{\mathrm{cris}}(\rho_v)$ one has the Hodge polygon $\mathrm{Hodge}(D)$, which is defined to be the convex hull of the points

$$\{(0,0)\} \cup \left\{ \left(i, \sum_{j=1}^{i} c_{2n+1-j} + (j-1) \right) : i = 1, \ldots, 2n \right\}$$

using the explicit recipe for the jumps in Hodge–Tate filtration in Theorem 3.4(3). One also has the Newton polygon $\mathrm{Newt}(D)$, which is defined to be the convex hull of the points

$$\{(0,0)\} \cup \left\{ \left(i, \sum_{j=1}^{i} v_p(\gamma_j) \right) : i = 1, \ldots, 2n \right\}.$$

Since D is weakly admissible, the Newton polygon lies on or above the Hodge polygon (and has equal endpoints). In particular, this implies that

$$v_p(\gamma_I) \geq \sum_{j=1}^{n} v_p(\gamma_j) \geq \sum_{j=1}^{n} c_{2n+1-j} + (j-1) = \sum_{j=1}^{n} c_{2n+1-j} + n(n-1)/2. \tag{7.3}$$

On the other hand, we can express the eigenvalues of $\mathcal{U}_S$ acting on π_p^J in terms of γ_I, using the results of [OST19, §4.1.1]. Indeed, note that we have $\pi_p^J = \pi_{0,p}^{J_0} \cong I(\chi)^{J_0}$ and, in the notation of *loc.cit.*, the action of the operator $[J\tau J]$ corresponds to the action of the operator $I_\chi(\mathbb{1}_{\check{\gamma}})$, with $\check{\gamma}(p) = \mathrm{diag}(p, \ldots, p, 1, \ldots, 1)$ (there are n lots of p). Therefore, one has an equality of polynomials

$$\det\left(1 - p^{-n^2/2}[J\tau J] \cdot X | \pi_p^J\right) = \det\left(1 - p^{-n^2/2} I_\chi(\mathbb{1}_{\check{\gamma}}) \cdot X | I(\chi)^{J_0}\right) = \prod_I (1 - \beta_I \cdot X)$$

where the product runs over all subsets $I \subset \{1, \ldots, 2n\}$ of size n, and $\beta_I = \prod_{i \in I} \beta_i$. This implies that the eigenvalues of $\mathcal{U}_S$ acting on π_p^J are of the form

$$p^\lambda p^{n^2/2} \beta_I = p^{\lambda + n^2/2 - n(n-1/2)} \gamma_I = p^{\lambda - n(n-1)/2} \gamma_I. \tag{7.4}$$

It turns out that, under Assumption 7.1(1), the algebraic representation with highest weight $(c_1, \ldots, c_{2n})$ is self-dual (see Lemma 7.5 below), which implies that $\lambda = -\sum_{j=1}^{n} c_{2n+1-j}$. Therefore, combining (7.3) and (7.4), it is reasonable (at least generically) to assume that $\mathcal{U}_S$ may have an eigenvalue which is a p-adic unit. In particular, if π_p is Borel ordinary (or equivalently, $\mathrm{Newt}(D) = \mathrm{Hodge}(D)$) then π_p is Siegel ordinary.

Let V^* be the algebraic representation of $\mathbf{G}_{\mathbb{C}}$ (which arises from an algebraic representation of $\mathbf{G}_E$) parameterising the discrete series L-packet of π_∞. By [BW13, §II.5], this implies that π is cohomological with respect to the representation V.

Lemma 7.5. *The representation V is self-dual.*

Proof. Let $\ell \notin S'$ be a split prime. Since $\pi_{0,f}$ admits a $\mathbf{H}_0(\mathbb{A}_f)$-linear model, the representation $\pi_{0,\ell}$ is $\mathbf{H}_0(\mathbb{Q}_\ell)$-distinguished, therefore by [JR96]

$$\pi_{0,\ell}^\vee \cong \pi_{0,\ell}.$$

This implies that π_0 is self-dual by the mild Chebotarev density theorem proven in [Ram15, Corollary B]. Since duality is preserved under base-change and we have assumed that any weak base-change Π of π is cuspidal, by strong multiplicity one (see [JS81, Theorem 4.4] for example) we must have

$$\Pi \cong \Pi^\vee.$$

In particular, this implies that V is self-dual since the algebraic representation of $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_{2n}$ corresponding to Π_0 has weight $(\mathbf{c}, -\mathbf{c}')$, where $\mathbf{c} = (c_1, \dots, c_{2n})$ is the weight of V^* and $\mathbf{c}' = (c_{2n}, \dots, c_1)$ (see Proposition 3.1). □

Let $\mathcal{R}$ be the set of all square-free products of rational primes that lie outside the set S' and split in the extension $E/\mathbb{Q}$ (we allow $1 \in \mathcal{R}$). For any positive integer m, let $\widehat{\mathcal{O}}_m^\times \subset \mathbb{A}_{E,f}^\times$ be the compact open subgroup consisting of the units of the profinite completion of $\mathcal{O}_m = \mathbb{Z} + m\mathcal{O}_E$, and let $D[m] \subset \mathbf{T}(\mathbb{A}_f)$ denote the image of $\widehat{\mathcal{O}}_m^\times$ under the map

$$\begin{array}{rcl} \mathscr{N} \colon \mathbb{A}_{E,f}^\times & \to & \mathbf{T}(\mathbb{A}_f) \\ z & \mapsto & \bar{z}/z \end{array}$$

By Lemma 2.3, $D[m]$ is a compact open subgroup. If we set $D_\ell = \mathscr{N}\left((\mathcal{O}_m \otimes \mathbb{Z}_\ell)^\times\right)$ for $\ell | m$, then $D[m]$ decomposes as $\prod_{\ell | m} D_\ell \times \prod_{\ell \nmid m} \mathscr{N}\left((\mathcal{O}_E \otimes \mathbb{Z}_\ell)^\times\right)$. Let $E[\infty]$ be the compositum of all ring class extensions $E[m]$ for $m \geq 1$ and let $\mathrm{Res}_{E[\infty]} \colon \mathrm{Gal}(E^{\mathrm{ab}}/E) \to \mathrm{Gal}(E[\infty]/E)$ denote the restriction map. Since the infinite place of E is complex, the restriction of the Artin map Art_E to the finite ideles is surjective onto the abelianisation of G_E.

Lemma 7.6. *There exists a continuous surjective homomorphism*

$$\mathrm{Art}_0 \colon \mathbf{T}(\mathbb{A}_f) \to \mathrm{Gal}(E[\infty]/E)$$

with kernel $\mathbf{T}(\mathbb{Q})$, which satisfies the following properties:

- *One has the following equality of compositions*

$$\mathrm{Res}_{E[\infty]} \circ \mathrm{Art} = \mathrm{Art}_0 \circ \mathscr{N}$$

- *For each $m \geq 1$, the map Art_0 induces an isomorphism $\mathbf{T}(\mathbb{A}_f)/\mathbf{T}(\mathbb{Q})D[m] \cong \mathrm{Gal}(E[m]/E)$.*
- *If $\ell \nmid m$ is split in E, then under the identification $\mathbf{T}(\mathbb{Q}_\ell) \cong \mathbb{Q}_\ell^\times$, the map Art_0 sends ℓ to the geometric Frobenius at λ in $\mathrm{Gal}(E[m]/E)$, where λ is the prime above ℓ distinguished by the fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$ (the identification is also with respect to this fixed embedding).*

Proof. It is well known (e.g. see [Kü17, Lemma 2]) that $E[\infty]$ is the fixed field of the transfer map $\mathrm{Ver} \colon G_{\mathbb{Q}}^{\mathrm{ab}} \to G_E^{\mathrm{ab}}$, so have a commuative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{A}_f^\times & \longrightarrow & \mathbb{A}_{E,f}^\times & \xrightarrow{\mathscr{N}} & \mathbf{T}(\mathbb{A}_f) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{Art}_{\mathbb{Q}}} & & \downarrow{\scriptstyle \mathrm{Art}_E} & & \downarrow{\scriptstyle \mathrm{Art}_0} & & \\ 1 & \longrightarrow & G_{\mathbb{Q}}^{\mathrm{ab}} & \xrightarrow{\mathrm{Ver}} & G_E^{\mathrm{ab}} & \xrightarrow{\mathrm{Res}} & \mathrm{Gal}(E[\infty]/E)) & \longrightarrow & 1 \end{array}$$

with exact rows, by Lemma 2.3. The surjectivity of Art_0 follows from that of Art_E. The isomorphism in the second assertion is induced from restricting the isomorphism $\mathrm{Gal}(E[m]/E) \cong \mathbb{A}_{E,f}^\times / E^\times \widehat{\mathcal{O}}_m^\times$ via $\mathscr{N}$ and Res. The third assertion follows by tracking the uniformiser under $\mathrm{Res} \circ \mathrm{Art}_E$. □

Let V be the (self-dual) representation of $\mathbf{G}_E$ with highest weight $\mathbf{c} = (0; c_1, \dots, c_{2n})$ associated with π as above, and let $\widetilde{V} := V \boxtimes \mathbf{1}$ denotes its trivial extension to $\widetilde{\mathbf{G}}_E$. We fix choices of highest weight vectors and lattices as in section 6.3. By abuse of notation, we let $\mathscr{V}$ (resp. $\widetilde{\mathscr{V}}$) denote both the relative Chow motive $\mathrm{Anc}_E(V)$ (resp. $\mathrm{Anc}_E(\widetilde{V})$) and the lisse $\mathbb{Q}_p$-sheaf $\mu(V_{\mathbb{Q}_p})$ (resp. $\mu(\widetilde{V}_{\mathbb{Q}_p})$). We also let $\mathscr{T}$ (resp. $\widetilde{\mathscr{T}}$) denote the equivariant sheaf associated with the representation T (resp. $\widetilde{T}$) as constructed in section 6.3. Since V is self-dual, the fixed parahoric vector in Assumption 7.1 gives rise to a "modular parametrisation", i.e. an equivariant linear map

$$v_\varphi \circ \mathrm{pr}_{\pi^\vee} \colon \mathrm{H}^{2n-1}_{\text{ét}}(\mathrm{Sh}_{\mathbf{G},\overline{\mathbb{Q}}}, \mathscr{V}(n)) \otimes \overline{\mathbb{Q}}_p \to W_\pi^*(1-n)$$

where v_φ is evaluation at φ. Since we have assumed that π is Siegel ordinary, this map factors through the ordinary part with respect to the operator $\mathcal{U}_S'$.

Definition 7.7. We let

$$T_\pi^*(1-n) := v_\varphi \circ \mathrm{pr}_{\pi^\vee}\left(M_{G,\text{ét},\mathbb{Z}_p}(K^pJ)\otimes \overline{\mathbb{Z}}_p\right) \subset W_\pi^*(1-n)$$

which is by construction a Galois stable $\overline{\mathbb{Z}}_p$-submodule inside $W_\pi^*(1-n)$. If Φ is a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}$, such that $W_\pi^*(1-n)$ and $v_\varphi \circ \mathrm{pr}_{\pi^\vee}$ are defined over Φ, then $T_\pi^*(1-n)$ is a Galois stable lattice with a model over $\mathcal{O}$ given by $v_\varphi \circ \mathrm{pr}_{\pi^\vee}\left(M_{G,\text{ét},\mathbb{Z}_p}(K^pJ)\otimes\mathcal{O}\right)$. Such an extension Φ always exists. We let $\mathrm{H}^1_{\mathrm{lf},\mathrm{Iw}}(E[mp^\infty], T_\pi^*(1-n))$ denote the locally finite Iwasawa cohomology (Definition A.13), and

$$\Psi_m : \mathrm{H}^1_{\mathrm{lf},\mathrm{Iw}}(E[mp^\infty], T_\pi^*(1-n)) \to \varprojlim_r \mathrm{H}^1(E[mp^r], W_\pi^*(1-n))$$

denote the natural maps. These maps are injective by Corollary A.19 and Assumption 3.5.

For every (sufficiently small) compact open subgroup $U \subset \mathbf{H}(\mathbb{A}_f)$, fix elements

$$\mathbf{1}_U \in \mathrm{H}^0_{\mathrm{mot}}(\mathrm{Sh}_{\mathbf{H}}(U), \mathbb{Z})$$

that are compatible under pull-backs, such that their common image $\mathbf{1} \in \mathrm{H}^0_{\mathrm{mot}}(\mathrm{Sh}_{\mathbf{H}}, \mathbb{Z})$ is fixed by the action of $\mathbf{H}(\mathbb{A}_f)$. By abuse of notation, we also let $\mathbf{1}_U$ denote the étale realisation in $\mathrm{H}^0_{\text{ét}}(\mathrm{Sh}_{\mathbf{H}}(U), \mathbb{Z}_p)$.

Definition 7.8. Let $m \in \mathcal{R}$ and $g \in \widetilde{G}^p$, and set $U_g = U_g^p \cdot \mathbf{H}(\mathbb{Z}_p)$ where

$$U_g^p := g\,(K^p \times D[m]^p)\,g^{-1} \cap H^p.$$

We define the following class

$$z_{g,m,\mathrm{Iw}} := v_\varphi \circ \mathrm{pr}_{\pi^\vee} \circ [g]_\star \circ \iota_{\star,\mathrm{Iw}}\left(\mathbf{1}_{U_g}\right) \in \varprojlim_r \mathrm{H}^1\left(E[mp^\infty], W_\pi^*(1-n)\right)$$

This definition extends by linearity to elements $z_{\phi,m,\mathrm{Iw}}$ for functions $\phi \in \mathcal{H}(\widetilde{G}^p)$ fixed by the action of $K^p \times D[m]^p$.

Lemma 7.9. *Let $m \in \mathcal{R}$ and ϕ as above. Then there exists a unique class in $\mathrm{H}^1_{\mathrm{lf},\mathrm{Iw}}(E[mp^\infty], T_\pi^*(1-n))$ that maps to $z_{\phi,m,\mathrm{Iw}}$ under Ψ_m.*

Proof. By construction, the class $z_{\phi,m,\mathrm{Iw}}$ is in the image of

$$\mathrm{H}^1_{\mathrm{Iw}}(E[mp^\infty], T_\pi^*(1-n)) \to \varprojlim_r \mathrm{H}^1(E[mp^r], W_\pi^*(1-n))$$

as $[g]_*$ and $\iota_{\star,\mathrm{Iw}}$ are pushforwards of $\mathbb{Z}_p$-valued cohomology functors and $T_\pi^*(1-n)$ was defined to be the (Galois-stable) image of values of these functors under the $\overline{\mathbb{Q}}_p$-linear maps v_φ and $\mathrm{pr}_{\pi^\vee}$. The fact that these classes are in the image of Ψ_m follows from our arguments in the proof of Lemma 6.17: the classes lift to classes in the cohomology of appropriate integral models of our Shimura–Deligne varieties, and therefore by Lemma A.14, are in the image of Ψ_m. As Ψ_m is injective by Corollary A.19, the result follows. ☐

Abusing notation, we will denote both the class in $\mathrm{H}^1_{\mathrm{Iw},\mathrm{lf}}(E[mp^\infty], T_\pi^*(1-n))$ and its image (under inflation) in $\mathrm{H}^1_{\mathrm{Iw}}(E[mp^\infty], T_\pi^*(1-n))$ by $z_{\phi,m,\mathrm{Iw}}$. The following proposition will allow us to prove norm compatibility relations for our Euler system classes.

Proposition 7.10. *Keeping with the same notation as in Definition 7.8, there exists a $\mathbf{H}(\mathbb{A}_f^p) \times \widetilde{\mathbf{G}}(\mathbb{A}_f^p)$-equivariant linear map*

$$\mathcal{L}_m \colon \mathcal{H}(\widetilde{G}^p)^{1\times D[m]^p} \longrightarrow \pi_f^\vee \otimes \varprojlim_r \mathrm{H}^1\left(E[mp^r], W_\pi^*(1-n)\right)$$

such that:

(a) *For any $\phi^{(m)} = \sum_i a_i \,\mathrm{ch}\,(g_i(K^p \times D[m]^p))$ in the Hecke algebra $\mathcal{H}(\widetilde{G}^p, \mathbb{Z}_p)$, the image of $z_{\phi^{(m)},m,\mathrm{Iw}}$ in the inverse limit $\varprojlim_r \mathrm{H}^1\left(E[mp^r], W_\pi^*(1-n)\right)$ satisfies*

$$z_{\phi^{(m)},m,\mathrm{Iw}} = v_\varphi \circ \mathcal{L}_m(\psi^{(m)})$$

where $\psi^{(m)} = \sum \frac{a_i}{\mathrm{Vol}(U_{g_i})}\,\mathrm{ch}\,(g_i(K^p \times D[m]^p))$.

(b) For a prime ℓ such that $\ell m \in \mathcal{R}$, one has

$$\operatorname{cores}^{E[\ell m p^\infty]}_{E[mp^\infty]} \mathcal{L}_{\ell m}(\psi^{(\ell m)}) = \mathcal{L}_m(\mathrm{pr}_*(\psi^{(\ell m)})).$$

where $\mathrm{pr}_* : \mathcal{H}(\widetilde{G}^p)^{1\times D[\ell m]^p} \to \mathcal{H}(\widetilde{G}^p)^{1\times D[m]^p}$ *is the natural pushforward.*

Proof. The map $\mathcal{L}_m$ is obtained by composing the completed pushforward of $\iota_{\star,\mathrm{Iw}}$ (see Proposition 5.11) with $\mathrm{pr}_{\pi^\vee}$ as follows. Since $1 \times D[m]^p$ lies in the centre of $\widetilde{G}^p$, the completed pushforward gives rise to a map

$$\hat{\iota}_{\star,\mathrm{Iw}} \colon \widehat{M}^p_{H,\mathbb{Q}_p} \otimes \mathcal{H}(\widetilde{G}^p)^{1\times D[m]^p} \longrightarrow \overline{N}_{\mathrm{Iw},\mathbb{Q}_p}(1 \times D[m]^p) \to \varprojlim_r \mathrm{H}^1\left(E[mp^r], \varinjlim_{L^p} M_{G,\text{ét},\mathbb{Q}_p}(JL^p)^{\mathrm{ord}}\right) \tag{7.11}$$

where the last map is the natural one. We define $\mathcal{L}_m$ to be the map

$$\mathcal{L}_m(\phi) := \mathrm{pr}_{\pi^\vee} \circ \widehat{\iota}_{\star,\mathrm{Iw}}(\mathbf{1} \otimes \phi).$$

The equivariance property for the map $\mathcal{L}_m$ follows from the equivariance of $\widehat{\iota}_{\star,\mathrm{Iw}}$ and the fact that the class $\mathbf{1}$ is fixed by the action of $\mathbf{H}(\mathbb{A}^p_f)$. Since the pushforward $\iota_{\star,\mathrm{Iw}}$ is compatible with base-change of coefficients, one obtains the required property from the commutative diagram in Corollary 5.12. This gives us (a). For claim (b), notice that there are induced maps $\overline{N}_{\mathrm{Iw},\mathbb{Q}_p}(1 \times D[\ell m]^p) \to \overline{N}_{\mathrm{Iw},\mathbb{Q}_p}(1 \times D[m]^p)$ coming from the pushforward of $N_{\mathrm{Iw},\mathbb{Q}_p}$ in the T^p component. By Lemma 6.24, the norm map corresponds the corestriction under the second morphism in (7.11). □

7.1. **The Euler system.** We continue with the same notation as in the previous sections. We consider the following test data for our Euler system: for $m \in \mathcal{R}$, define an element $\phi^{(m)} \in \mathcal{H}(\widetilde{G}^p, \mathbb{Z})^{K^p \times D[m]^p}$ as the product

$$\phi^{(m)} := \left(\bigotimes_{\substack{\ell \notin S \\ \ell \nmid m}} \phi_{\ell,0}\right) \otimes \left(\bigotimes_{\ell \mid m} \phi_\ell\right) \otimes \phi_S$$

where

- $\phi_S \in \mathcal{H}(\widetilde{G}_S, \mathbb{Z})^{K_S \times \mathscr{N}((\mathcal{O}_E \otimes \mathbb{Z}_S)^\times)}$ is fixed.
- $\phi_{\ell,0} = \mathrm{ch}\,(K_\ell \times \mathbf{T}(\mathbb{Z}_\ell)) \in \mathcal{H}(\widetilde{G}_\ell, \mathbb{Z})$ (note that $\mathscr{N}((\mathcal{O}_E \otimes \mathbb{Z}_\ell)^\times) = \mathbf{T}(\mathbb{Z}_\ell)$ for primes ℓ that do not ramify in $E/\mathbb{Q}$, by the proof of Lemma 2.3).
- Using the notation in Theorem 4.1, we define $\phi_\ell = \sum_{i=0}^n a_i \,\mathrm{ch}\,((g_i,1)(K_\ell \times D_\ell))$, where the element g_i is
$$g_i = 1 \times \begin{pmatrix} 1 & \ell^{-1}X_i \\ & 1 \end{pmatrix} \quad \in \quad \mathrm{GL}_1(\mathbb{Q}_\ell) \times \mathrm{GL}_{2n}(\mathbb{Q}_\ell) = \mathbf{G}(\mathbb{Q}_\ell).$$
The integers a_i are defined to be
$$a_i := (\ell - 1) \cdot b_i \cdot [H(\mathbb{Z}_\ell) : V_{1,i}]^{-1}$$
where $V_{1,i} = (g_i,1)(K_\ell \times D_\ell)(g_i,1)^{-1} \cap H_\ell$ (this quantity a_i is an integer by part (3) of *loc.cit.*).

Definition 7.12. Let π_0 be a cuspidal automorphic representation of $\mathbf{G}_0$ satisfying Assumption 7.1. For $m \in \mathcal{R}$, we define the *Iwasawa Euler system class* to be

$$\mathcal{Z}_{m,\mathrm{Iw}} := m^{-n^2} \cdot z_{\phi^{(m)},m,\mathrm{Iw}} \quad \in \quad \mathrm{H}^1_{\mathrm{Iw}}\left(E[mp^\infty], T^*_\pi(1-n)\right)$$

which are well-defined elements by Lemma 7.9.

We now prove our main result of this article.

Theorem 7.13. *Let $\ell, m \in \mathcal{R}$ such that ℓ is prime and their product satisfies $\ell m \in \mathcal{R}$. The Iwasawa Euler system classes satisfy*

$$\operatorname{cores}^{E[\ell m p^\infty]}_{E[mp^\infty]} \mathcal{Z}_{\ell m,\mathrm{Iw}} = P(\mathrm{Fr}_\lambda^{-1}) \cdot \mathcal{Z}_{m,\mathrm{Iw}} \tag{7.14}$$

where $P(X) = \det\left(1 - \mathrm{Frob}_\lambda^{-1} X | \rho_\pi(n)\right)$ and λ is the unique prime of E lying above ℓ fixed by the embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$.

Proof. We will perform a series of reduction steps to reduce the proof to that of Theorem 4.1.

Step 1. Since the classes $\mathcal{Z}_{m,\mathrm{Iw}}$ are elements of the submodule $\mathrm{H}^1_{\mathrm{lf},\mathrm{Iw}}(E[mp^\infty], T_\pi^*(1-n))$ and the maps Ψ_m are injective by Corollary A.19, it suffices to verify the claimed relations in $\varprojlim_r \mathrm{H}^1(E[mp^r], W_\pi^*(1-n))$ considered as a $\overline{\mathbb{Q}}_p$-vector space. By Proposition 7.10 (b), there exist a commutative diagram

$$\begin{array}{ccc} \mathcal{H}(\widetilde{G}^p)^{1\times D[\ell m]^p} & \xrightarrow{\mathcal{L}_{\ell m}} & \pi_f^\vee \otimes \varprojlim_r \mathrm{H}^1\left(E[\ell m p^r], W_\pi^*(1-n)\right) \\ \downarrow{\scriptstyle \mathrm{pr}_*} & & \downarrow{\scriptstyle \mathrm{cores}^{E[\ell m p^\infty]}_{E[mp^\infty]}} \\ \mathcal{H}(\widetilde{G}^p)^{1\times D[m]^p} & \xrightarrow{\mathcal{L}_m} & \pi_f^\vee \otimes \varprojlim_r \mathrm{H}^1(E[mp^r], W_\pi^*(1-n)) \end{array}$$

and functions $\psi^{(\ell m)}$ and $\psi^{(m)}$ such that:

- $\mathcal{Z}_{\ell m,\mathrm{Iw}} = \ell^{-n^2} m^{-n^2} (v_\varphi \circ \mathcal{L}_{\ell m})(\psi^{(\ell m)})$
- $\mathcal{Z}_{m,\mathrm{Iw}} = m^{-n^2} (v_\varphi \circ \mathcal{L}_m)(\psi^{(m)})$.

By the diagram above, and cancelling the m^{-n^2} factor, this reduces the theorem to proving the statement

$$(v_\varphi \circ \mathcal{L}_m)\left(\mathrm{pr}_*(\psi^{(\ell m)})\right) = \ell^{n^2} P(\mathrm{Fr}_\lambda^{-1}) \cdot (v_\varphi \circ \mathcal{L}_m)\left(\psi^{(m)}\right). \tag{7.15}$$

We note that both functions $\mathrm{pr}_*(\psi^{(\ell m)})$ and $\psi^{(m)}$ decompose as $\mathrm{pr}_*(\psi_\ell^{(\ell m)}) \otimes \psi^\ell$ and $\psi_\ell^{(m)} \otimes \psi^\ell$ for some fixed element $\psi^\ell \in \mathcal{H}(\widetilde{G}^{\ell p})$, where

$$\begin{aligned} \left(\psi^{(\ell m)}\right)_\ell &= \sum_i \frac{a_i}{\mathrm{Vol}(V_{1,i})} \mathrm{ch}\left((g_i, 1)(K_\ell \times \mathbf{T}(\mathbb{Z}_\ell))\right) \quad \in \quad \mathcal{H}(\widetilde{G}_\ell) \\ \psi_\ell^{(m)} &= \frac{1}{\mathrm{Vol}\left((K_\ell \times \mathbf{T}(\mathbb{Z}_\ell)) \cap H_\ell\right)} \mathrm{ch}(K_\ell \times \mathbf{T}(\mathbb{Z}_\ell)) \quad \in \quad \mathcal{H}(\widetilde{G}_\ell) \end{aligned}$$

using the explicit formulae in Proposition 7.10(a) and at the start of section 7.1.

Step 2. We now simplify the expression in (7.15) further by cutting out by a finite-order character. Since it is enough to prove the relation (7.15) at each finite layer of the inverse limit $\varprojlim_r \mathrm{H}^1\left(E[mp^r], W_\pi^*(1-n)\right)$, it is enough to show the relation in (7.15) after post-composing both sides by any Galois equivariant morphism

$$\varprojlim_r \mathrm{H}^1\left(E[mp^r], W_\pi^*(1-n)\right) \to \overline{\mathbb{Q}}_p(\chi) \tag{7.16}$$

where χ is a finite-order character of $\mathrm{Gal}(E[mp^\infty]/E)$.

More precisely, let χ be a finite-order character of $\mathrm{Gal}(E[mp^\infty]/E)$. Since the Galois action and the action of $\mathbf{T}(\mathbb{A}_f^p) = T^p$ are intertwined by the character κ^{-1} (see Lemma 6.24), we can view χ as a character of T^p by setting $\chi(t) = \chi(\sigma)$, where $\sigma \in \mathrm{Gal}(E[mp^\infty]/E)$ is an element such that $\kappa(\sigma^{-1})$ equals the image of t in $\varprojlim_r \Delta(D[m]^p D_{p^r})$. This character factorises as $\chi_\ell \cdot \chi^\ell$, where χ_ℓ is a character of T_ℓ and χ^ℓ is a character of $T^{\ell p} = \mathbf{T}(\mathbb{A}_f^{\ell p})$.

Furthermore, the Frobenius element Fr_λ^{-1} satisfies $\kappa(\mathrm{Fr}_\lambda^{-1}) = \ell^{-1}$, so the action of $P(\mathrm{Fr}_\lambda^{-1})$ is intertwined with multiplication by

$$P(\chi_\ell(\ell)) = P_\lambda(\chi_\ell(\ell)\ell^{-n}) = L(\pi_\ell \otimes \chi_\ell, 1/2)^{-1}$$

under the map in (7.16), where $P_\lambda(X)$ is as in Theorem 3.4.

Let $\mathcal{L}_m^\chi$ denote the composition of $\mathcal{L}_m$ with the map $1\otimes$(7.16). This is a $\mathbf{H}(\mathbb{A}_f^p) \times \widetilde{\mathbf{G}}(\mathbb{A}_f^p)$-equivariant linear map $\mathcal{H}(\widetilde{G}^p) \to (\pi_f \boxtimes \chi^{-1})^\vee$, and to prove the theorem, we are required to prove the statement

$$(v_\varphi \circ \mathcal{L}_m^\chi)\left(\psi^{(\ell m)}\right) = \ell^{n^2} L(\pi_\ell \otimes \chi_\ell, 1/2)^{-1} \left(v_\varphi \circ \mathcal{L}_m^\chi\right)\left(\psi^{(m)}\right) \tag{7.17}$$

for all finite-order characters χ.

Step 3. We now reduce the statement in (7.17) to one in local representation theory. The map

$$\begin{aligned} \mathfrak{Z} \colon \mathcal{H}(\widetilde{G}_\ell)^{1\times D[m]^p} &\longrightarrow (\pi_\ell \boxtimes \chi_\ell^{-1})^\vee \\ \upsilon &\mapsto \left(v_{\varphi^\ell} \circ \mathcal{L}_m^\chi\right)(\upsilon \otimes \psi^\ell) \end{aligned}$$

where $v_{\varphi^\ell}\colon (\pi_f^\ell \boxtimes (\chi^\ell)^{-1})^\vee \otimes (\pi_\ell \boxtimes \chi_\ell^{-1})^\vee \to (\pi_\ell \boxtimes \chi_\ell^{-1})^\vee$ is the evaluation map at the vector $\varphi^\ell = \bigotimes_{q \notin S' \cup \{\ell\}} \varphi_q \otimes \varphi_S \otimes \varphi_p$, is $H_\ell \times \widetilde{G}_\ell$-equivariant by construction. Note that we have

$$(\ell-1)^{-1} \cdot \mathrm{Vol}\left((K_\ell \times \mathbf{T}(\mathbb{Z}_\ell)) \cap H_\ell\right) \cdot \frac{a_i}{\mathrm{Vol}(V_{1,i})} = b_i$$

where b_i are the integers appearing at the start of section 7.1. So by multiplying the relation in (7.17) by $(\ell-1)^{-1} \cdot \mathrm{Vol}\left((K_\ell \times \mathbf{T}(\mathbb{Z}_\ell)) \cap H_\ell\right)$ and using the explicit formulae in Step 1, we are reduced to proving the relation:

$$(7.18) \quad (v_{\varphi_\ell} \circ \mathfrak{Z}) \left(\sum_i b_i \,\mathrm{ch}\left((g_i,1)(K_\ell \times \mathbf{T}(\mathbb{Z}_\ell))\right) \right) = \frac{\ell^{n^2}}{\ell-1} L(\pi_\ell \otimes \chi_\ell, 1/2)^{-1} \cdot (v_{\varphi_\ell} \circ \mathfrak{Z}) \left(\mathrm{ch}(K_\ell \times \mathbf{T}(\mathbb{Z}_\ell))\right).$$

Step 4. We now apply Frobenius reciprocity. Let v denote the element $\sum_i b_i \,\mathrm{ch}\left((g_i,1)(K_\ell \times \mathbf{T}(\mathbb{Z}_\ell))\right)$. By Lemma 5.10 (b), the map $\mathfrak{Z}$ corresponds to an element $\mathfrak{z} \in \mathrm{Hom}_H(\pi_\ell \boxtimes \chi_\ell^{-1}, \mathbb{C})$ (implicitly using the identification of $\overline{\mathbb{Q}}_p$ and $\mathbb{C}$), and the relation in (7.18) takes the form

$$\mathfrak{z}(v \cdot \varphi_\ell) = \frac{\ell^{n^2}}{\ell-1} L(\pi_\ell \otimes \chi_\ell, 1/2)^{-1} \cdot \mathfrak{z}(\varphi_\ell)$$

noting that φ_ℓ is spherical. This relation now follows from Theorem 4.1 (using the fact that χ_ℓ is unramfied) and this completes the proof of Theorem 7.13. □

Definition 7.19. We define the *anticyclotomic Euler system class at level mp^r*, denoted c_{mp^r}, to be the image of $\mathcal{Z}_{m,\mathrm{Iw}}$ under the natural map

$$\mathrm{H}^1_{\mathrm{Iw}}\left(E[mp^\infty], T_\pi^*(1-n)\right) \longrightarrow \mathrm{H}^1\left(E[mp^r], T_\pi^*(1-n)\right).$$

As a corollary of Theorem 7.13, we immediately obtain Theorem A in the introduction with the classes c_{mp^r}.

7.2. **Motivic interpretation.** It will come as no surprise to the reader that the Euler system classes c_{mp^r} arise from elements in the motivic cohomology (see section 1.2) of $\mathrm{Sh}_{\widetilde{\mathbf{G}}}$. More precisely, we consider the following definition.

Definition 7.20. Let $g \in \widetilde{\mathbf{G}}(\mathbb{A}_f)$ and $L \subset \widetilde{\mathbf{G}}(\mathbb{A}_f)$ a compact open subgroup. Set $U := gLg^{-1} \cap \mathbf{H}(\mathbb{A}_f)$. Then we define

$$\mathcal{Z}_{g,\mathrm{mot}} := ([g]_\star \circ (\iota,\nu)_\star \circ \mathrm{br})(\mathbf{1}_U) \in \mathrm{H}^{2n}_{\mathrm{mot}}\left(\mathrm{Sh}_{\widetilde{\mathbf{G}}}(L), \widetilde{\mathscr{V}}(n)\right)$$

where $\widetilde{\mathscr{V}} = \mathrm{Anc}_{\widetilde{\mathbf{G}},E}(\widetilde{V})$ as before. This definition can be extended by linearity to elements $\mathcal{Z}_{\phi,\mathrm{mot}}$ where $\phi \in \mathcal{H}(\widetilde{\mathbf{G}}(\mathbb{A}_f))^L$.

Unwinding the definition of the classes $\mathcal{Z}_{m,\mathrm{Iw}}$ we have the following result.

Proposition 7.21. *For $r \geq 1$, set $\phi = \phi^{(m)} \otimes \mathrm{ch}\left(\eta_r(J \times D_{p^r})\right)$ where η_r is the element defined in the proof of Proposition 6.22. Then the image of (c_{mp^∞}) in $\varprojlim_r \mathrm{H}^1(E[mp^r], W_\pi^*(1-n))$ is equal to the image of $\mathcal{Z}_{\phi,\mathrm{mot}}$ under the following composition:*

$$\begin{aligned}
\varprojlim_r \mathrm{H}^{2n}_{\mathrm{mot}}\left(\mathrm{Sh}_{\widetilde{\mathbf{G}}}(K^pJ \times D[m]^pD_{p^r}), \widetilde{\mathscr{V}}(n)\right) &\xrightarrow{r_{\text{ét}}} \varprojlim_r \mathrm{H}^{2n}_{\text{ét}}\left(\mathrm{Sh}_{\widetilde{\mathbf{G}}}(K^pJ \times D[m]^pD_{p^r}), \widetilde{\mathscr{V}}(n)\right)^{\mathrm{lf}} \\
&\xrightarrow{(m^{-n^2}(\mathcal{U}_S')^{-r})\cdot} \varprojlim_r \mathrm{H}^{2n}_{\text{ét}}\left(\mathrm{Sh}_{\widetilde{\mathbf{G}}}(K^pJ \times D[m]^pD_{p^r}), \widetilde{\mathscr{V}}(n)\right)^{\mathrm{fs}} \\
&\xrightarrow{e_{\mathrm{ord}} \circ \mathrm{AJ}_{\mathrm{Iw}}} \varprojlim_r \mathrm{H}^1\left(E[mp^r], \mathrm{H}^{2n-1}_{\text{ét}}\left(\mathrm{Sh}_{\mathbf{G}}(K^pJ), \mathscr{V}(n)\right)^{\mathrm{ord}}\right) \\
&\xrightarrow{\mathrm{pr}_{\pi^\vee}} \pi_f^\vee \otimes \varprojlim_r \mathrm{H}^1\left(E[mp^r], W_\pi^*(1-n)\right) \\
&\xrightarrow{v_\varphi} \varprojlim_r \mathrm{H}^1\left(E[mp^r], W_\pi^*(1-n)\right)
\end{aligned}$$

where $\mathrm{AJ}_{\mathrm{Iw}}$ *denotes the Abel–Jacobi map in Proposition A.10,* $\mathcal{U}'_S$ *is the Hecke operator introduced in Assumption 7.1, and the "ordinary/finite-slope parts" are with respect to* $\mathcal{U}'_S$ *(recall that* $\mathcal{U}'_S$ *is intertwined with the action of* $\mathcal{T}$ *on the integral cohomology).*

The fact that these classes come from motivic cohomology will allow us to investigate the image of our Euler system classes in other cohomology theories, e.g. in Deligne cohomology or syntomic cohomology as defined by Nekovář and Nizioł ([NN16]). We will investigate these questions in a future paper. We end with the following result showing that, under suitable conditions, the anticyclotomic Euler system classes actually live in the Bloch–Kato Selmer group (after inverting p).

Proposition 7.22. *Let* π_0 *be a cuspidal automorphic representation satisfying Assumption 7.1 and let* $S_{\mathrm{ns}} \subset S$ *denote the subset of all primes which are either inert or ramified in* $E/\mathbb{Q}$*. Then the image of the anticyclotomic Euler system classes* c_{mp^r} *in* $\mathrm{H}^1(E[mp^r], W_\pi^*(1-n))$ *are unramified at all primes* λ *of* $E[mp^r]$ *which do not lie above a prime in* $S_{\mathrm{ns}} \cup \{p\}$*, i.e. the image of* c_{mp^r} *under the map*

$$\mathrm{H}^1(E[mp^r], W_\pi^*(1-n)) \to \mathrm{H}^1(E[mp^r]_\lambda, W_\pi^*(1-n)) \to \mathrm{H}^1(I_\lambda, W_\pi^*(1-n))$$

is trivial, where $I_\lambda \subset G_{E[mp^r]}$ *denotes the inertia subgroup at* λ*.*

Furthermore, for any prime $\mathfrak{P}$ *of* $E[mp^r]$ *dividing* p*, the restriction of the class* c_{mp^r} *at* $\mathfrak{P}$ *lies in the Bloch–Kato local group* $\mathrm{H}^1_f(E[mp^r]_{\mathfrak{P}}, W_\pi^*(1-n))$*.*

Proof. It is enough to prove the statement for $r \geq 1$. In this case, if λ is a prime not dividing a rational prime in

$$\widetilde{S} := S \cup \{ \text{ primes dividing } mp^r \}$$

then the class c_{mp^r} is constructed from sub-Shimura varieties which have hyperspecial level outside $\widetilde{S}$. By [Lan13], these Shimura varieties have smooth integral models over $\mathcal{O}_E[S^{-1}]$, which implies that the class c_{mp^r} is unramified at λ.

If λ is a prime lying above $\ell \notin S_{\mathrm{ns}} \cup \{p\}$ and ℓ splits in $E/\mathbb{Q}$, then the decomposition group of λ in $E[mp^\infty]/E[mp^r]$ is infinite. Indeed, it is enough to check that the prime of E lying below λ doesn't split completely in the anticyclotomic $\mathbb{Z}_p$-extension of E, and this is standard. In this case the result follows from [Rub00, Corollary B.3.5], since the classes c_{mp^r} are universal norms in this extension.

The comparison of syntomic cohomology and étale cohomology in [NN16] and the fact that the classes come from motivic classes imply that, at primes dividing p, the classes c_{mp^r} lie in the local group H^1_g. A criterion for when H^1_g and H^1_f coincide is that p^{-1} is not an eigenvalue of the crystalline Frobenius φ on $\mathbf{D}_{\mathrm{cris}}(W_\pi^*(1-n))$. By part (3) of Theorem 3.4 the eigenvalues of φ are given by $p^{n-1}\alpha_i^{-1}$ where

$$L(\Pi_v, s + (1-2n)/2)^{-1} = \prod_{i=1}^{2n}(1 - \alpha_i p^{-s}).$$

Now α_i are Weil numbers of weight $2n-1$, because the representation Π satisfies the Ramanujan–Petersson conjecture (see [Mor10, Corollary 8.4.9]). This implies that the eigenvalues of φ are Weil numbers of weight -1, hence $p^{n-1}\alpha_i^{-1}$ cannot equal p^{-1}. □

Corollary 7.23. *Suppose that the weak base-change* Π *is unramified outside a finite set of primes that split in* $E/\mathbb{Q}$*. Then*

$$c_{mp^r} \in \mathrm{H}^1_f\left(E[mp^r], W_\pi^*(1-n)\right)$$

i.e. the classes live in the Bloch–Kato Selmer group after inverting p*.*

Proof. Let $\mathfrak{L}$ be a prime of $E[mp^r]$ lying above a prime ℓ that ramifies in $E/\mathbb{Q}$, and let λ denote the prime of E lying below $\mathfrak{L}$. By assumption, the representation Π_λ is unramified, so [CH13, Theorem 3.2.3] implies that the representation ρ_π is unramified at λ and the eigenvalues $\beta_1, \ldots, \beta_{2n}$ of $\mathrm{Frob}_\lambda^{-1}$ acting on $\rho_\pi^*(1-n)$ satisfy

$$L(\Pi_\lambda, s + (1-2n)/2)^{-1} = \prod_{i=1}^{2n}\left(1 - \beta_i^{-1}(\mathrm{Nm}\,\lambda)^{s+n-1}\right) = \prod_{i=1}^{2n}\left(1 - \beta_i^{-1}\ell^{s+n-1}\right).$$

As in the proof of Proposition 7.22, the quantities $\ell^{n-1}\beta_i^{-1}$ are Weil numbers of weight $2n-1$, and since $\mathrm{Frob}_{\mathfrak{L}}$ is a power of Frob_λ, we see that the eigenvalues of $\mathrm{Frob}_{\mathfrak{L}}$ acting on $\rho_\pi^*(1-n)$ cannot possibly equal 1. This implies that

$$\mathrm{H}^0\left(E[mp^r]_{\mathfrak{L}}, W_\pi^*(1-n)\right) = 0. \tag{7.24}$$

Since $\Pi \cong \Pi^\vee \cong \Pi^c$ we have $W_\pi(n) \cong W_\pi^*(1-n)$ (see the proof of Lemma 7.5 and Theorem 3.4). Combining this with (7.24) implies

$$\mathrm{H}^1\left(E[mp^r]_{\mathfrak{L}}, W_\pi^*(1-n)\right) = 0$$

so in particular, any local condition at $\mathfrak{L}$ is vacuous. The result then follows from Proposition 7.22. □

Remark 7.25. The conditions in Corollary 7.23 are completely analogous to the case of Heegner points. Indeed, in this setting one considers an elliptic curve whose conductor is divisible only by primes that split in the imaginary quadratic extension.

Appendix A. Continuous étale cohomology

In this appendix, we explicate in a general setting two constructions in Jannsen's continuous étale cohomology that we have made use of in this article. The first of these are the "pushforward functors" – compositions of certain Gysin and trace morphisms. In [KLZ17, §2.1], these were defined in cases when continuous étale cohomology coincides with the usual definition of (p-adic) étale cohomology. The second is an Iwasawa theoretic version of the (p-adic) étale Abel–Jacobi map that allows us to "bypass" cohomological triviality when the p-adic Lie extension along which the limit is taken has positive dimension. In §A.3, we record some results on torsion in the Iwasawa cohomology of such p-adic Lie extensions.

A.1. **Pushforwards maps.** For any abelian category $\mathcal{A}$, we let $D^+(\mathcal{A})$ denote the derived category of left-bounded complexes in $\mathcal{A}$, and if $\mathcal{C}$ is any category, we denote by $\mathcal{C}^{\mathbb{N}}$ the category of inverse systems of objects in $\mathcal{C}$. If $\mathcal{C}$ is Grothendieck abelian, so is $\mathcal{C}^{\mathbb{N}}$, in which case $D^+(\mathcal{C}^{\mathbb{N}}) = D^+(\mathcal{C})^{\mathbb{N}}$. Moreover, if $F : \mathcal{C} \to \mathcal{D}$ is a left exact functor, then so is $F^{\mathbb{N}} : \mathcal{C}^{\mathbb{N}} \to \mathcal{D}^{\mathbb{N}}$, and if $\mathcal{C}$ has enough injectives, so does $\mathcal{C}^{\mathbb{N}}$, in which case the right derived functors of F and $F^{\mathbb{N}}$ are related by $R^i(F^{\mathbb{N}}) = (R^iF)^{\mathbb{N}}$ [Jan88, Proposition 1.2]. If X is a scheme, we denote by $\mathbf{Sh}(X_{\text{ét}})$ the category of étale sheaves of abelian groups on the small étale site $X_{\text{ét}}$. We let $\Gamma_X : \mathbf{Sh}(X_{\text{ét}})^{\mathbb{N}} \to \mathbf{Ab}$ denote the functor $\mathscr{F} = (\mathscr{F}_n) \mapsto \varprojlim_n H^0(X, \mathscr{F}_n)$. Then, following [Jan88, §3], the continuous étale cohomology of $\mathscr{F}$ is defined to be

$$\mathrm{H}^j_{\text{ét}}(X, \mathscr{F}) := R^j\Gamma_X(\mathscr{F}).$$

Lemma A.1. *Let X, Y be quasi-compact, quasi-separated (qcqs) schemes. For any quasi-finite separated morphism $f\colon X \to Y$, there exists a pair of functors*

$$f_!\colon \mathbf{Sh}(X_{\text{ét}})^{\mathbb{N}} \to \mathbf{Sh}(Y_{\text{ét}})^{\mathbb{N}} \qquad f^!\colon \mathbf{Sh}(Y_{\text{ét}})^{\mathbb{N}} \to \mathbf{Sh}(X_{\text{ét}})^{\mathbb{N}}$$

satisfying the following properties:

(1) $f_!$ is exact and $f_! \dashv f^!$ i.e. for any $\mathscr{F} \in \mathbf{Sh}(X_{\text{ét}})^{\mathbb{N}}$ and $\mathscr{G} \in \mathbf{Sh}(Y_{\text{ét}})^{\mathbb{N}}$, there are functorial isomorphisms

$$\mathrm{Hom}(f_!\mathscr{F}, \mathscr{G}) \xrightarrow{\sim} \mathrm{Hom}(\mathscr{F}, f^!\mathscr{G}).$$

(2) If $f = gh$ with g, h both quasi-finite and separated, then there are canonical isomorphisms $f_! \cong g_! \circ h_!$ and $f^! \cong h^! \circ g^!$. Moreover, these isomorphisms are compatible with any third such composition.

(3) $f_!$ is a sub-functor of f_* and if f is proper, $f_! = f_*$.

(4) $f^!$ is a sub-functor of f^* and if f is etale, $f^! = f^*$.

Moreover, these functors induce the usual derived functors

$$f_!\colon D^+(\mathbf{Sh}(X_{\text{ét}})^{\mathbb{N}}) \to D^+(\mathbf{Sh}(Y_{\text{ét}})^{\mathbb{N}}) \qquad Rf^!\colon D^+(\mathbf{Sh}(Y_{\text{ét}})^{\mathbb{N}}) \to D^+(\mathbf{Sh}(X_{\text{ét}})^{\mathbb{N}}).$$

Proof. See Proposition 3.1.4 and 3.1.8 in [AGV73, Exp. XVIII] and Proposition 6.1.4 and Theorem 5.1.8 in [AGV73, Exp. XVII]. In a more general setting, these are also found in [Sta20, Tag 0F4U]. □

We will be interested in applying these functors to the following sub-class of sheaves.

Definition A.2. Let X be a qcqs scheme. An *étale $\mathbb{Z}_p$-sheaf*, or a *p-adic étale sheaf* $\mathscr{F} \in \mathbf{Sh}(X)^{\mathbb{N}}_{\text{ét}}$ is an inverse system $\{\mathscr{F}_n\}_{n\geq 1}$ where

- $\mathscr{F}_n$ is a constructible $\mathbb{Z}/p^n\mathbb{Z}$-module on $X_{\text{ét}}$,

- the transition maps $\mathscr{F}_{n+1} \to \mathscr{F}_n$ induce isomorphisms $\mathscr{F}_{n+1} \otimes_{(\mathbb{Z}/\ell^{n+1}\mathbb{Z})} (\mathbb{Z}/\ell^n\mathbb{Z}) \cong \mathscr{F}_n$.

We say that $\mathscr{F}$ is *lisse* or that $\mathscr{F}$ is a *local system* if each $\mathscr{F}_n$ is locally constant. A morphism of such sheaves is just a morphism of these objects in $\mathbf{Sh}(X_{\text{ét}})^{\mathbb{N}}$.

Definition A.3. Let X, Y be schemes, $\mathscr{F} \in \mathbf{Sh}(X_{\text{ét}})^{\mathbb{N}}$ and $\mathscr{G} \in \mathbf{Sh}(Y_{\text{ét}})^{\mathbb{N}}$. A *pushforward* $(f, \phi)_\star \colon (X, \mathscr{F}) \to (Y, \mathscr{G})$ is a morphism $f \colon X \to Y$ of schemes and a morphism $\phi : \mathscr{F} \to f^*\mathscr{G}$ of sheaves on X.

Definition A.4. Let S be a scheme. An *étale smooth S-pair of codimension c* is a morphism $f \colon X \to Y$ of smooth qcqs S-schemes satisfying the following condition: there exists a smooth S-scheme $\bar{Y}$ and a factorization $f : X \xrightarrow{h} \bar{Y} \xrightarrow{g} Y$ such that h is a closed immersion with fibers over each point of S of pure codimension c in $\bar{Y}$ and g is étale. If $f' \colon X' \to Y'$ is another such pair, then a *morphism* from f to f' is a pair of étale maps $p \colon X' \to X$ and $q \colon Y' \to Y$ that commute with f, f'.

Proposition A.5. *Let $f \colon X \to Y$ be an étale smooth S-pair of codimension c and let $\mathscr{F}$, $\mathscr{G}$ be lisse étale $\mathbb{Z}_p$-sheaves on X, Y respectively. Assume that p is invertible on S. Then for any pushforward $(f, \phi)_* : (X, \mathscr{F}) \to (Y, \mathscr{G})$ and any $j \in \mathbb{Z}_{\geq 0}$, there is an induced "pushforward" on cohomology*

$$(f, \phi)_* \colon \mathrm{H}^j_{\text{ét}}(X, \mathscr{F}) \to \mathrm{H}^{j+2c}_{\text{ét}}(Y, \mathscr{G}(c))$$

that is functorial and Cartesian:

- *(Functoriality) if $f' \colon X' \to Y'$ is another such pair and $(p, q) \colon f \to f'$ is a morphism, then we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{H}^j_{\text{ét}}(X', p^*\mathscr{F}) & \xrightarrow{(f',\psi)_*} & \mathrm{H}^{j+2c}_{\text{ét}}(Y', q^*\mathscr{G}(c)) \\ \Big\downarrow{\scriptstyle \mathrm{Tr}_p} & & \Big\downarrow{\scriptstyle \mathrm{Tr}_q} \\ \mathrm{H}^j_{\text{ét}}(X, \mathscr{F}) & \xrightarrow{(f,\phi)_*} & \mathrm{H}^{j+2c}_{\text{ét}}(Y, \mathscr{G}(c)) \end{array}$$

  where $\psi = p^*\phi$.
- *(Cartesian) if $q : Y' \to Y$ is any finite étale morphism, $X' = X \times_Y Y'$, $f' : X' \to Y'$, $p : X' \to X$ are the natural morphisms, then we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{H}^j_{\text{ét}}(X', p^*\mathscr{F}) & \xrightarrow{(f',\psi)_*} & \mathrm{H}^{j+2c}_{\text{ét}}(Y', q^*\mathscr{G}(c)) \\ \Big\uparrow{\scriptstyle p^*} & & \Big\uparrow{\scriptstyle q^*} \\ \mathrm{H}^j_{\text{ét}}(X, \mathscr{F}) & \xrightarrow{(f,\phi)_*} & \mathrm{H}^{j+2c}_{\text{ét}}(Y, \mathscr{G}(c)) \end{array}$$

  where $\psi = p^*\phi$.

When f is a closed immersion, $(f, \phi)_\star$ is defined in [Jan88, Theorem 3.17]. The proof below is along similar lines except that we account for the independence of the factorization of f.

Proof. Let $f : X \xrightarrow{h} \bar{Y} \xrightarrow{g} Y$ be a factorization. As g is étale, $g^! = g^*$ by Lemma A.1 (4). By [Jan88, Eqn. 3.20], $h^!\mathbb{Z}_p(c) = \mathbb{Z}_p[-2c]$. Thus, by equation (3.19) of *op.cit*, $h^!\mathscr{G}(c) = h^*\mathscr{G} \otimes h^!\mathbb{Z}_p(c) = h^*\mathscr{G}[-2c]$. Therefore,

$$Rf^!\mathscr{G}(c) = Rg^! \circ Rh^! \mathscr{G}(c) = g^*h^*\mathscr{G}[-2c] = f^*\mathscr{G}[-2c]$$

Deriving $\mathrm{Tr}_f \colon \Gamma_X f^! \to \Gamma_Y$, we obtain a natural transformation $R(\mathrm{Tr}_f) : R(\Gamma_X f^!) = R\Gamma_X \circ Rf^! \to R\Gamma_Y$ between the two derived functors. Evaluating $R(\mathrm{Tr}_f)$ at $\mathscr{G}^*(c)$, we obtain an induced map $\mathrm{H}^j_{\text{ét}}(X, f^*\mathscr{G}) \cong R^{j+2c}(\Gamma_X f^!)(\mathscr{G}(c)) \xrightarrow{\mathrm{Tr_f}} \mathrm{H}^{j+2c}_{\text{ét}}(Y, \mathscr{G}(c))$ via passage to cohomology. The pushforward map is now defined to be

$$(f, \phi)_* = \mathrm{H}^j_{\text{ét}}(X, \mathscr{F}) \xrightarrow{\phi} \mathrm{H}^j_{\text{ét}}(X, f^*\mathscr{G}) \xrightarrow{\mathrm{Tr}_f} \mathrm{H}^{j+2c}_{\text{ét}}(Y, \mathscr{G}(c)).$$

Lemma A.1 (ii) then gives the independence of this map on the choice of $\bar{Y}$. The functoriality of these pushforwards follows from that of the trace, i.e. $R\mathrm{Tr}_f \circ R\mathrm{Tr}_p = R\mathrm{Tr}_q \circ R\mathrm{Tr}_{f'}$. The Cartesian property follows by applying proper base change ([Sta20, Tag 095S]) to $q : Y' \to Y$. ☐

A.2. **Abel–Jacobi map in Iwasawa cohomology.** Let E be a field of characteristic 0, and let $\mathbf{FtSch}_{\text{ét}}$ be the category of schemes of finite-type over E with morphisms given by étale maps. Let X be such a scheme and $\mathscr{F}$ a lisse étale $\mathbb{Z}_p$-sheaf on X. We denote by $\overline{X}$ the base change $X \times_{\operatorname{Spec} E} \operatorname{Spec} \bar{E}$. Set

$$M^i(X) := \mathrm{H}^i_{\text{ét}}(X, \mathscr{F}), \qquad \overline{M}^i(X) := \mathrm{H}^i_{\text{ét}}(\overline{X}, \mathscr{F}),$$

where $\mathrm{H}^i_{\text{ét}}$ is continuous étale cohomology as defined in the previous subsection.

Theorem A.6 (Jannsen, Deligne)**.** *There is a first quadrant cohomological spectral sequence*

$$E_2^{i,j} \colon \mathrm{H}^i(E, \overline{M}^j(X)) \Rightarrow M^{i+j}(X)$$

which we refer to as the Hochschild–Serre spectral sequence.

We say an element in $M^i(X)$ is *cohomologically trivial* if it is in the kernel of the natural base-change map $M^i(X) \to \mathrm{H}^0(E, \overline{M}^i(X))$. We write $M^i(X)_0$ for the subspace of all cohomologically trivial elements in $M^i(X)$.

Definition A.7 (Abel–Jacobi map)**.** Let $\mathrm{AJ}_{\text{ét}}$ denote the edge map

$$\mathrm{AJ}_{\text{ét}} \colon M^i(X)_0 \longrightarrow \mathrm{H}^1\left(E, \overline{M}^{i-1}(X)\right)$$

arising from the Hochschild–Serre spectral sequence.

For any finite extension F/E, set $X_F = X \times_{\operatorname{Spec} E} \operatorname{Spec} F$ and consider it as an E-scheme. Then $\overline{X_F} = \bigsqcup_\tau X_{\overline{E}}$ where τ runs over the elements of $\operatorname{Hom}_E(F, \overline{E})$, therefore

$$\overline{M}^i(X_F) = M^i(\overline{X_F}) = \mathbb{Z}_p[\Delta_F] \otimes_{\mathbb{Z}_p} \overline{M}^i(X) \tag{A.8}$$

where $\Delta_F = \operatorname{Hom}_E(F, \overline{E})$ is considered as a $\operatorname{Gal}(\overline{E}/E)$-set (with the left action given by post-composition). If F'/F is a finite extension, then the natural map $\mathrm{pr} \colon X_{F'} \to X_F$ induces two maps on cohomology, namely

$$\begin{aligned} \mathrm{pr}_\star \colon M^i(X_{F'}) \to M^i(X_F), \qquad & \mathrm{pr}^\star \colon M^i(X_F) \to M^i(X_{F'}) \\ \mathrm{pr}_\star \colon \overline{M}^i(X_{F'}) \to \overline{M}^i(X_F), \qquad & \mathrm{pr}^\star \colon \overline{M}^i(X_F) \to \overline{M}^i(X_{F'}). \end{aligned}$$

In the geometric case, these maps are the ones induced by

$$\begin{aligned} \mathrm{pr}_\star \colon \Delta_{F'} \twoheadrightarrow \Delta_F, \qquad & \mathrm{pr}^\star \colon \Delta_F \hookrightarrow \Delta_{F'} \\ \tau \mapsto \tau_{|F} \qquad & S \mapsto \sum_{\eta_{|F} = S} \eta \end{aligned}$$

Fix a pro-system $(\Gamma_r)_{r \geq 1}$ of finite groups and let $\Gamma = \Gamma_\infty = \varprojlim_r \Gamma_r$. Let $\mathbf{\Gamma} = \mathbf{\Gamma}_E$ be the category defined as follows:

- the objects of $\mathbf{\Gamma}$ are Γ-towers $E \subset F_0 \subset F_1 \subset F_2 \subset \dots$ contained in $\overline{E}$, i.e. F_r/E are finite Galois for all $r \geq 0$, and $\Gamma_r = \operatorname{Gal}(F_r/F_0)$ for $r \geq 1$.
- the morphisms of $\mathbf{\Gamma}$ are reverse inclusions of towers, i.e. a morphism $(F'_r)_r \to (F_r)_r$ is a system of inclusions $F_r \hookrightarrow F'_r$ for $r \geq 0$ compatible with the inclusion morphisms of the two systems.

We let F_∞ denote the compositum of F_r. Then F_∞/E is Galois, and $\operatorname{Gal}(F_\infty/F_0) \cong \Gamma$. If we let Γ_0 be the trivial group and let $\Gamma^r := \ker(\Gamma \to \Gamma_r)$ for $r \geq 0$, then $(\Gamma^r)_{r \geq 0}$ forms decreasing (exhaustive and separated) filtration of open normal subgroups of Γ and we may equivalently describe objects of $\mathbf{\Gamma}$ as towers $(F_r)_{r \geq 0}$ Galois over E such that $\operatorname{Gal}(F_\infty/F_r) = \Gamma^r$ for $r \geq 0$. We have a distinguished embedding $\Gamma = \operatorname{Gal}(F_\infty/F_0) \hookrightarrow \operatorname{Gal}(F_\infty/E)$, and thus a third description of objects of $\mathbf{\Gamma}$ is as Galois extensions F_∞/E contained in $\overline{E}$ that come with an embedding of the *filtered* group Γ as an open normal subgroup of their Galois group, and the morphisms $F'_\infty \to F_\infty$ are now inclusions $F_\infty \hookrightarrow F'_\infty$ such that the induced morphism $\operatorname{Gal}(F'_\infty/E) \to \operatorname{Gal}(F_\infty/E)$ induces the identity map on Γ. We will follow this perspective in the sequel, and abusively say that $F_\infty \in \mathbf{\Gamma}$ is an object.

Example A.1. Let p be an odd prime and $\Gamma_r = \mathbb{Z}_p^\times/(1 + p^r\mathbb{Z}_p)$ for $r \geq 1$. Then $\Gamma = \mathbb{Z}_p^\times$. Let E be an imaginary quadratic number field in which p splits, and let m be a positive integer satisfying $(m, p) = 1$. For $r \geq 1$, we let $F_r = E[mp^r]$ denote the ring class field of conductor mp^r. Then $(F_r)_r \in \mathbf{\Gamma}$. If $m|m'$ and $F'_\infty = E[m'p^\infty]$, we obtain a morphism $F'_\infty \to F_\infty$.

Definition A.9. For $X \in \mathbf{FtSch}_{\text{ét}}$, $F_\infty \in \mathbf{\Gamma}$, denote $X \times_{\operatorname{Spec} E} \operatorname{Spec} F_r$ by X_r. We set

- $\mathrm{H}^i_{\mathrm{Iw}}(F_\infty, \overline{M}^j(X)) := \varprojlim_r \mathrm{H}^i(F_r, M^j(\overline{X}))$,
- $M^i_{\mathrm{Iw}}(F_\infty, X) := \varprojlim_r M^i(X_r)$,
- $\overline{M}^i_{\mathrm{Iw}}(F_\infty, X) := \varprojlim_r M^i(\overline{X}_r)$.

Here, the first limit, known as the *Iwasawa cohomology of* $\overline{M}^j(X)$, is with respect to corestriction, and other two are with respect to $\mathrm{pr}_\star$. By (A.8), the last limit is also equal to $\mathbb{Z}_p[\![\mathrm{Gal}(F_\infty/E)]\!] \otimes_{\mathbb{Z}_p} \overline{M}^i(X)$. These are then covariant (resp. contravariant) bifunctors (i.e. functors in each variable) on $\mathbf{\Gamma} \times \mathbf{FtSch}_{\text{ét}}$ with respect to pushforwards (resp. pullbacks).

Proposition A.10. *Suppose that p^∞ divides the profinite order of Γ. Then for any $F_\infty \in \mathbf{\Gamma}$, $X \in \mathbf{FtSch}_{\text{ét}}$, there is an induced "Abel-Jacobi map"*

$$\mathrm{AJ}_{\mathrm{Iw}} \colon M^i_{\mathrm{Iw}}(F_\infty, X) \to \mathrm{H}^1_{\mathrm{Iw}}(F_\infty, \overline{M}^{i-1}(X)).$$

that is bifunctorial with respect to both pushforwards and pullbacks in each variable, i.e. $\mathrm{AJ}_{\mathrm{Iw}}$ *is a natural transformation of the (four) bi-functors on* $\mathbf{FtSch}_{\text{ét}} \times \mathbf{\Gamma}$.

Proof. By Shapiro's lemma, we know that $\mathrm{H}^i(E, \overline{M}^j(X_r)) = \mathrm{H}^i\left(E, \mathbb{Z}_p[\Delta_{F_r}] \otimes \overline{M}^j(X)\right) = \mathrm{H}^i(F_r, \overline{M}^j(X))$ for each r, with pushforwards corresponding to corestriction. The exact sequences $0 \to M^i(X_r)_0 \to M^i(X_r) \to \mathrm{H}^0(F_r, \overline{M}^i(X))$ therefore form an inverse system, and so we get

$$0 \to M^i_{\mathrm{Iw}}(F_\infty, X)_0 \to M^i_{\mathrm{Iw}}(F_\infty, X) \to \mathrm{H}^0_{\mathrm{Iw}}(F_\infty, \overline{M}^i(X))$$

where $M^i_{\mathrm{Iw}}(F_\infty, X)_0$ is defined as the inverse limit of $M^i(X_r)_0$. As $\overline{M}^i(X)$ is of finite type over $\mathbb{Z}_p$ ([Del77, Corollaire 1.10, Chapitre 7]), we have $\mathrm{H}^0_{\mathrm{Iw}}(F_\infty, \overline{M}^i(X)) = 0$ by [Rub00, Lemma 3.2, Appendix B], so we see that $M^i_{\mathrm{Iw}}(F_\infty, X)_0 = M^i_{\mathrm{Iw}}(F_\infty, X)$. As we also have an inverse system of Abel–Jacobi maps

$$\mathrm{AJ}_{\text{ét}} \colon M^i(X_r)_0 \longrightarrow \mathrm{H}^1\left(F_r, \overline{M}^{i-1}(X)\right)$$

for each level, we obtain the result by passing to the limit. □

A.3. **Torsion in Iwasawa cohomology.** Let p be an odd prime and G a profinite group. We say that G satisfies condition (F) if:

- G has finite p-cohomological dimension.
- For any finite discrete $\mathbb{F}_p[G]$-module T, and any integer $i \geq 0$, $\mathrm{H}^i(G, T)$ is finite.

If G satisfies condition (F), then the continuous Galois cohomology groups $\mathrm{H}^i(G, -)$ are finitely-generated (see [Nek06, Proposition 4.2.3]). We record the following well-known lemma which will be used in the latter half of this section.

Lemma A.11. *Let F be a number field and Σ a finite set of places of F containing all primes above p and all archimedean places. Let F_Σ denote the maximal extension of F which is unramified outside Σ. Then $\mathrm{Gal}(F_\Sigma/F)$ satisfies condition (F).*

Proof. See [NSW08, Proposition 8.3.18, Theorem 8.3.20]. □

Let E be a number field and Φ a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and uniformiser π. Let T be a free $\mathcal{O}$-module of finite rank with a continuous action of $\mathrm{Gal}(\overline{E}/E)$ such that T is ramified at only a finite set of places of E. We set $W := T \otimes_{\mathcal{O}} \Phi$ and $A := W/T$ the divisible Galois representation. Let $\mathbf{\Gamma}$ be the category as above and let $\mathbf{\Gamma}^{\text{fin}}$ be the sub-category of extensions F_∞ such that only finitely many primes of $F_0 = (F_\infty)^\Gamma$ are ramified in F_∞. When $\Gamma = \mathbb{Z}_p$ or $\mathbb{Z}_p^\times$, we have $\mathbf{\Gamma} = \mathbf{\Gamma}^{\text{fin}}$ [NSW08, Proposition 11.1.1].

Let $F_\infty \in \mathbf{\Gamma}^{\text{fin}}$ and set $F = F_0$. For any set of places Σ of F, we let F_Σ denote the maximal extension of F unramified away from Σ. Let $S(F)$ denote the finite set of places of F containing all primes above p, all archimedean places, all places where T is ramified and all places ramified in F_∞. Then for any finite set of places $\Sigma \supset S(F)$, F_Σ contains F_∞ and T is a $\mathrm{Gal}(F_\Sigma/F_r)$-module for all $r \geq 0$, $r = \infty$.

Definition A.12. Let $\Sigma \supset S(F)$ be as above. We define the following cohomology groups for $i = 0, 1, 2$.

- The *unramified away from Σ-cohomology* of T is defined to be
$$\mathrm{H}^i_\Sigma(F,T) := \mathrm{H}^i(\mathrm{Gal}(F_\Sigma/F),T).$$
- The *unramified away from Σ-Iwasawa cohomology* (or just *Σ-Iwasawa cohomology*) of T is defined to be
$$\mathrm{H}^i_{\Sigma,\mathrm{Iw}}(F_\infty,T) := \varprojlim_r \mathrm{H}^i(\mathrm{Gal}(F_\Sigma/F_r),T)$$
with limit taken with respect to corestriction.

If $\Sigma_1 \subset \Sigma_2$ are two such sets, then there are induced inflation morphisms $\mathrm{Inf}_{\Sigma_1,\Sigma_2} : \mathrm{H}^i_{\Sigma_1}(F,T) \to \mathrm{H}^i_{\Sigma_2}(F,T)$. As inflation commutes with corestriction, there are also induced maps $\mathrm{Inf}_{\Sigma_1,\Sigma_2} : \mathrm{H}^i_{\Sigma_1,\mathrm{Iw}}(F_\infty,T) \to \mathrm{H}^i_{\Sigma_2,\mathrm{Iw}}(F_\infty,T)$

Definition A.13. The *locally finite Iwasawa cohomology* of T is defined to be the inductive limit
$$\mathrm{H}^i_{\mathrm{lf},\mathrm{Iw}}(F_\infty,T) = \varinjlim_\Sigma \mathrm{H}^i_{\Sigma,\mathrm{Iw}}(F_\infty,T).$$
over all Σ with respect to the induced inflations maps. When $i=1$, it is equal to the union of the images of all $\mathrm{Inf}_\Sigma : \mathrm{H}^1_{\Sigma,\mathrm{Iw}}(F_\infty,T) \hookrightarrow \mathrm{H}^1_{\mathrm{Iw}}(F_\infty,T)$.

The above groups are relevant in light of the following result.

Lemma A.14. *Let X be smooth E-scheme and $\mathscr{F} \in \mathbf{Sh}(X_{\text{ét}})^{\mathbb{N}}$ a lisse étale $\mathbb{Z}_p$-sheaf. Suppose that for some finite set of places Σ of F, there exist*
- *a projective system of smooth $\mathcal{O}_{F,\Sigma}$-schemes $(\mathcal{X}_r)_r$ with Galois transition morphisms and Galois group isomorphic to $(\Gamma_r)_r$,*
- *maps $p : X \times_{\operatorname{Spec} F} \operatorname{Spec} F_r \to \mathcal{X}_r$ of $\operatorname{Spec} \mathcal{O}_E$-schemes such that $X_r \to \operatorname{Spec} F$ is the pullback of $\mathcal{X}_r \to \mathcal{O}_{F,\Sigma}$ under p.*
- *lisse étale $\mathbb{Z}_p$-sheaves $\mathfrak{F}_r$ on $\mathcal{X}_r$ compatible under pullbacks on the projective system such that $\mathscr{F}$ on X_r is the pullback of $\mathfrak{F}_r$ along p.*

Suppose moreover that p^∞ divides the profinite order of Γ. Then the image of
$$\varprojlim_r \mathrm{H}^i_{\text{ét}}(\mathcal{X}_r,\mathfrak{F}_r) \xrightarrow{p^\star} M^i_{\mathrm{Iw}}(F_\infty,X) \xrightarrow{\mathrm{AJ}_{\mathrm{Iw}}} \mathrm{H}^1_{\mathrm{Iw}}(F_\infty,\overline{M}^{i-1}(X))$$
is contained in $\mathrm{H}^1_{\Sigma,\mathrm{Iw}}(F_\infty,\overline{M}^{i-1}(X))$.

Proof. This follows from [Jan88, Theorem 3.3], and functoriality of the Hochschild-Serre spectral sequence under morphisms of schemes. □

For the rest of this section, our general reference is [NSW08, Chapter V] and [Jan89].

Definition A.15. The *Iwasawa algebra* of Γ, denoted $\Lambda = \Lambda(\Gamma) = \Lambda_{\mathcal{O}}(\Gamma) = \mathcal{O}[\![\Gamma]\!]$ is defined to be the completed group ring $\varprojlim_r \mathcal{O}[\Gamma_r]$. We equip $\Lambda(\Gamma)$ with the projective limit topology induced by the π-adic topology of $\mathcal{O}[\Gamma_r]$ which makes $\Lambda(\Gamma)$ a profinite ring. A left *topological module* over $\Lambda(\Gamma)$ is a Hausdorff abelian topological group M with a continuous action $\Lambda(\Gamma) \times M \to M$. A right module is defined similarly. Via the involution $\Gamma \to \Gamma$, $\gamma \mapsto \gamma^{-1}$, one can view a right module as a left module. If M is also locally compact, we let $M^\vee = \mathrm{Hom}_{\mathrm{cts}}(M,\mathbb{Q}_p/\mathbb{Z}_p)$ denote its Pontrjagin dual, which carries a natural $\Lambda(\Gamma)$-module structure.

If Γ is a compact p-adic Lie group, then $\Lambda(\Gamma)$ is (left and right) Noetherian [Ven02, §1.2]. Moreover, if Γ is pro-p, then $\Lambda(\Gamma)$ is a local ring [NSW08, Proposition 5.2.16 (iii)] and if Γ is torsion free, then $\Lambda(\Gamma)$ is an integral domain, i.e. it does not have zero divisors [Ven02, §1.2].

Definition A.16. Let M be a topological $\Lambda(\Gamma)$-module. The (generalized) *Iwasawa adjoints* of M are defined to be
$$E^i(M) := \mathrm{Ext}^i_\Lambda(M,\Lambda)$$
which are (right) Λ-modules via functoriality and the right-module structure of Λ. We denote the Λ-dual $E^0(M) = \mathrm{Hom}_\Lambda(M,\Lambda)$ by M^+. Then, there is a canonial map $\varphi_M \colon M \to M^{++}$ and the *torsion submodule* of M is defined to be the kernel of φ_M.

If M is finitely generated and Γ is a p-adic Lie group, then the torsion can equivalently be described as the set of torsion elements (in the ring theoretic sense) in M over the (necessarily integral) Iwasawa algebra $\Lambda(\Gamma')$ of an open pro-p, p-torsion free subgroup $\Gamma' \subset \Gamma$. [Ven02, §2]. In particular, Λ and M^+ are Λ-torsion free, as is any sub-module of M^+.

Proposition A.17 (Jannsen)**.** *Suppose Γ is a compact p-adic Lie group. For any $\Sigma \supset S(F)$, there is a first quadrant cohomological spectral sequence of finitely generated Λ-modules*

$$E_2^{p,q} = E^p(\mathrm{H}^q\left(\mathrm{Gal}(F_\Sigma/F_\infty), A)^\vee\right) \Rightarrow \mathrm{H}^{p+q}_{\Sigma,\mathrm{Iw}}(F_\infty, T)$$

Proof. See [Jan14, Theorem 1] (or [TV17, Theorem 8.3] and the reference within). The finiteness claim uses Lemma A.11. □

Corollary A.18. *Suppose $E_2^{1,0} = 0$. Then we have a canonical injection*

$$\mathrm{H}^1_{\Sigma,\mathrm{Iw}}(F_\infty, T) \hookrightarrow \left(\mathrm{H}^1(\mathrm{Gal}(F_\Sigma/F_\infty), V/T)^\vee\right)^+.$$

In particular, if $\Gamma = \mathbb{Z}_p^\times$ and $T^{\mathrm{Gal}(\overline{E}/F_\infty)} = 0$, then $\mathrm{H}^1_{\Sigma,\mathrm{Iw}}(F_\infty, T)$ is Λ-torsion free.

Proof. The spectral sequence gives rise to the following 5-term exact sequence

$$0 \to E_2^{1,0} \to \mathrm{H}^1_{\Sigma,\mathrm{Iw}}(F_\infty, T) \to E_2^{0,1} \to E_2^{2,0} \to H^2_{\Sigma,\mathrm{Iw}}(F_\infty, T)$$

from which the first claim follows. If $\Gamma = \mathbb{Z}_p^\times$, and $T^{\mathrm{Gal}(\overline{E}/F_\infty)} = T^{\mathrm{Gal}(F_\Sigma/F_\infty)} = 0$, then $\mathrm{H}^0(\mathrm{Gal}(F_\Sigma/F_\infty), A))^\vee$ is finite. By [Jan89, Proposition 2.6(ii)], $E_2^{1,0} = 0$ and hence the final claim. □

Corollary A.19. *Let Γ be the filtered group $\mathbb{Z}_p^\times$ (with filtration $\Gamma^r = 1 + p^r\mathbb{Z}_p$ for $r \geq 1$) and suppose that W is absolutely irreducible as a $\mathrm{Gal}(\overline{E}/E)$-representation with $\dim_\Phi W \geq 2$. Then for any $F_\infty \in \mathbf{\Gamma}$ with F_∞ an abelian extension of E, the canonical map*

$$\Psi : \mathrm{H}^1_{\mathrm{lf},\mathrm{Iw}}(F_\infty, T) \to \varprojlim_r \mathrm{H}^1(F_r, W)$$

is injective.

Proof. First notice that $\mathrm{H}^0(E^{\mathrm{ab}}, T) = 0$. Suppose for the sake of contradiction that $v \in T \subset W$ is a non-zero vector fixed by $\mathrm{Gal}(\overline{E}/E^{\mathrm{ab}})$. As W is irreducible, the $\mathrm{Gal}(\overline{E}/E)$-span of any non-zero vector (in particular, v) must be all of W. But this implies that the action of $\mathrm{Gal}(\overline{E}/E)$ on W factors through $\mathrm{Gal}(E^{\mathrm{ab}}/E)$, and hence $W \otimes_\Phi \overline{\mathbb{Q}}_p$ contains a one-dimensional subrepresentation – a contradiction.

By Corollary A.18, it therefore suffices to show that the kernel of Ψ is contained in the union over all Σ of the Λ-torsion submodules (in the sense of Definition A.16) of $\mathrm{H}^1_{\mathrm{lf},\Sigma}(F_\infty, T) \subset \mathrm{H}^1_{\mathrm{lf},\mathrm{Iw}}(F_\infty, T)$. Equivalently, it suffices to show that $\ker \Psi$ is contained in the union of the $\Lambda(\Gamma^1) \cong \Lambda(\mathbb{Z}_p)$-torsion sub-modules (in the ring theoretic sense) of $\mathrm{H}^1_{\mathrm{lf},\Sigma}(F_\infty, T)$. Clearly $\ker \Psi$ contains the $\mathcal{O}$-torsion of $\mathrm{H}^1_{\mathrm{lf},\mathrm{Iw}}(F_\infty, T)$, which is $\Lambda(\Gamma^1)$-torsion, so it suffices to verify that for all $\Sigma \supset S(F)$, the kernel of

$$\Psi_\Sigma : \mathrm{H}^1_{\Sigma,\mathrm{Iw}}(F_\infty, T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \varprojlim_r \mathrm{H}^1(F_r, W)$$

is contained in the $R := \Lambda(\Gamma^1)[1/p]$-torsion. To this end, set $M := \mathrm{H}^1_{\Sigma,\mathrm{Iw}}(F_\infty, T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Then $M = \varprojlim_r M_{\Gamma^r}$ where M_U denotes the $\Phi[\Gamma/U]$-module of U-coinvariants for a compact open subgroup $U \subset \Gamma$, and moreover Ψ_Σ is given by the inverse limit of the induced maps $\Psi_r : M_{\Gamma^r} \to \mathrm{H}^1(F_r, W)$. The remainder of this proof follows [LLZ14, Proposition A.2.1, Proposition A.2.6].

By Shapiro's lemma (see [Nek06, §8.2]), we have

$$M = \mathrm{H}^1_\Sigma(F, \Lambda(\Gamma) \otimes_{\mathcal{O}} T)[1/p] \qquad \mathrm{H}^1(F_r, W) = \mathrm{H}^1(F, \mathcal{O}[\Gamma_r] \otimes_{\mathcal{O}} T)[1/p]$$

with appropriate actions of $\mathrm{Gal}(F_\Sigma/F)$ and $\mathrm{Gal}(\overline{F}/F)$ on $\Lambda(\Gamma) \otimes_{\mathcal{O}} T$ and $\mathcal{O}[\Gamma_r] \otimes_{\mathcal{O}} T$ respectively. Let $(\gamma_r)_{r\geq 1}$ denote a compatible choice of topological generators for $(\Gamma^r)_{r\geq 1}$. One has a short exact sequence of $\Lambda(\Gamma)[\mathrm{Gal}(F_\Sigma/F)]$-modules

$$0 \to \Lambda(\Gamma) \otimes_{\mathcal{O}} T \xrightarrow{[\gamma_r]-1} \Lambda(\Gamma) \otimes_{\mathcal{O}} T \to \mathcal{O}[\Gamma_r] \otimes_{\mathcal{O}} T \to 0.$$

After passing to the long exact sequence on cohomology and inverting p we see that, for $r \geq 1$, the homomorphism Ψ_r is the composition of an injective homomorphism and inflation (which is injective on H^1). Therefore the map Ψ_r is injective for $r \geq 1$. This implies that the kernel of Ψ_Σ is contained in

$$\bigcap_{r\geq 1}([\gamma_r]-1)M \subset M_{\mathrm{tors}}$$

where M_{tors} is the R-torsion submodule (in the ring-theoretic sense), and the last inclusion follows from the fact that M is finitely generated over $R \cong \mathcal{O}[\![X]\!][1/p]$ (by Lemma A.11 and [Nek06, Proposition 4.2.3]), which is a PID (so we can write M as the direct sum of a free module and a torsion module). □

Appendix B. Shimura–Deligne varieties

In this article, we have worked with varieties that are strictly speaking not Shimura varieties in the sense of [Del79]. Specifically, axiom 2.1.1.3 of *op.cit.*, which we will refer to as (SD3), fails for the morphism $h\colon \mathbb{S} \to \mathbf{H}$ considered in §2.2 (since $\mathbf{H}^{\mathrm{ad}}$ has a compact $\mathbb{Q}$-simple factor $\mathrm{SU}(n)$). More generally, for a Shimura datum arising from PEL moduli problems, (SD3) usually does not hold, e.g. see [Mor10, §1.1], or [Lan12, Remark 2.5.8]. The primary reason for imposing it is that it allows one to apply strong approximation, which is used in describing the reciprocity law on the geometric connected components [Del71, §3.4]. Another application is in reducing the problem of the existence of canonical models to the case of connected Shimura varieties [Del79, §2.7]. Assuming (SD3) however excludes the case of so-called Shimura sets, for example the Gross curve in [Zha14, 3.1], which are instrumental in applications to Euler systems.

The purpose of this appendix is to record results that continue to hold for datum that do not necessarily satisfy (SD3). We make no claim of originality here as most proofs carry over verbatim. The proofs that we have chosen to include are primarily for purposes of exposition of the originals. We shall freely use results from [Del71] when they do not invoke the hypothesis in §2.1 where (SD3) assumed.

Notation. We fix an algebraic closure $\mathbb{C}$ of $\mathbb{R}$, and take $\overline{\mathbb{Q}}$ to be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. We denote the Deligne torus by $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ and let $w\colon \mathbb{G}_{m,\mathbb{R}} \to \mathbb{S}$ defined via the *inverse* of the inclusion $\mathbb{R}^\times \hookrightarrow \mathbb{C}^\times$. We fix the identification $\mathbb{S}_{\mathbb{C}} \cong \mathbb{G}_m \times \mathbb{G}_m$ such that the inclusion $\mathbb{S}(\mathbb{R}) \to \mathbb{S}(\mathbb{C})$ is given by $z \mapsto (z, \overline{z})$, and we take $\mu\colon \mathbb{G}_{m,\mathbb{C}} \to \mathbb{S}_{\mathbb{C}}$ to be the cocharacter $z \mapsto (z, 1)$. For an algebraic group $\mathbf{G}$, we denote by $\mathbf{Z_G}$ its centre, $\mathbf{G}^{\mathrm{der}}$ its derived group, and $\mathbf{G}^{\mathrm{ad}}$ its quotient by $\mathbf{Z_G}$. A superscript "+" (e.g. $\mathbf{G}(\mathbb{R})^+$) denotes the connected component of the identity in the analytic topology.

B.1. **Preliminaries.**

Definition B.1. A *Shimura–Deligne datum* is a pair $(\mathbf{G}, X)$ consisting of a connected reductive algebraic group $\mathbf{G}$ over $\mathbb{Q}$ and a $\mathbf{G}(\mathbb{R})$-conjugacy class X of homomorphisms $h\colon \mathbb{S} \to \mathbf{G}_{\mathbb{R}}$ satisfying

(SD1) For all $h \in X$, the Hodge bigrading of the complex vector space $\mathrm{Lie}(\mathbf{G})_{\mathbb{C}}$ under the adjoint action of $\mathbb{S}_{\mathbb{C}}$ is of type $\{(-1,1),(0,0),(1,-1)\}$. In particular, the cocharacter $h \circ w : \mathbb{G}_m \to \mathbf{G}_{\mathbb{R}}$ is central and independent of h.

(SD2) For any $h \in X$, $\mathrm{ad}\left(h(\sqrt{-1})\right) : \mathbf{G}_{\mathbb{R}} \to \mathbf{G}_{\mathbb{R}}$ (which is an involution by (SD1)) is a Cartan involution of $\mathbf{G}^{\mathrm{der}}_{\mathbb{R}}$, i.e. the real Lie group

$$\left\{g \in \mathbf{G}^{\mathrm{der}}(\mathbb{C}) \,|\, h(\sqrt{-1})\overline{g}h(\sqrt{-1})^{-1} = g\right\}$$

is compact.

A *morphism* $(\mathbf{G}_1, X_1) \to (\mathbf{G}_2, X_2)$ of Shimura–Deligne datum is a homomorphism $u\colon \mathbf{G}_1 \to \mathbf{G}_2$ such that $u(X_1) \subset X_2$. We say that such a morphism is *injective* if u is.

Remark B.2. These are the axioms (1.5.1), (1.5.2), (1.5.3) of [Del71], or (2.1.1.1), (2.1.1.2) of [Del79]. We have borrowed the above terminology from [GN09].

Remark B.3. It suffices to verify the axiom for one $h \in X$, and we may write $(\mathbf{G}, h)$ for the Shimura–Deligne datum. Notice also that since $\mathbf{G}^{\mathrm{der}} \to \mathbf{G}^{\mathrm{ad}}$ is a (central) isogeny, (SD2) is equivalent to requiring that $\mathrm{ad}\left(h(\sqrt{-1})\right)$ induces a Cartan involution of $\mathbf{G}^{\mathrm{ad}}_{\mathbb{R}}$.

Fix $h_0 \in X$ and let K_∞ be the centraliser of h_0 in $\mathbf{G}(\mathbb{R})$, so that $X = \mathbf{G}(\mathbb{R})/K_\infty$. Then K_∞ contains the centre of $\mathbf{G}(\mathbb{R})$ and $\mathrm{Lie}(K_\infty)_{\mathbb{C}}$ coincides with $\mathrm{Lie}(\mathbf{G})^{(0,0)}$. Since $h_0(\sqrt{-1})$ acts as -1 on $\mathrm{Lie}(\mathbf{G})/\mathrm{Lie}(K_\infty)$, we see that h_0 is central if and only if $h_0(\sqrt{-1})$ is. Since Cartan involutions are unique up to conjugacy, this is equivalent to $\mathbf{G}^{\mathrm{der}}(\mathbb{R})$ being compact. Such a pair will be referred to as a *trivial Shimura–Deligne datum.*

Let X^0 be the connected component of X containing h_0. By [Del79, Corollary 1.1.17], X^0 is either a singleton (which happens if and only if $(\mathbf{G}, h_0)$ is trivial) or a Hermitian symmetric domain. More precisely, if $\mathbf{G}^{\mathrm{ad}}_{\mathbb{R}} = \mathbf{G}_1 \times \dots \mathbf{G}_k$ is the decomposition into $\mathbb{R}$-simple factors, then $X^0 = X_1 \times \cdots \times X_k$ where each X_i is a quotient of $\mathbf{G}_i(\mathbb{R})^+$ by a maximal compact subgroup.

Lemma B.4. *The Lie group $K'_\infty = K_\infty \cap \mathbf{G}^{\mathrm{der}}(\mathbb{R})^+$ is connected, and $K_\infty = \mathbf{Z}(\mathbb{R}) \cdot K'_\infty$.*

Proof. As K'_∞ is a maximal compact subgroup in a connected Lie group, it is connected. Let $\mathbf{G}^*$ denote the compact real form of $\mathbf{G}^{\mathrm{ad}}_{\mathbb{R}}$ defined by $h(\sqrt{-1})$. Let $h'_0\colon \mathbb{S} \to \mathbf{G}^{\mathrm{ad}}$ be the induced map, and let $\mathbf{C}$ be the centraliser of h'_0 in $\mathbf{G}^{\mathrm{ad}}_{\mathbb{R}}$. Then $\mathbf{C}$ is a Cartan subgroup of $\mathbf{G}^{\mathrm{ad}}$, and is therefore (Zariski) connected. Moreover, $\mathbf{C}(\mathbb{R})$ is a closed subgroup of $\mathbf{G}^*(\mathbb{R})$, so $\mathbf{C}$ is $\mathbb{R}$-anisotropic and hence $\mathbf{C}(\mathbb{R})$ is connected by [BT65, 14.3]. As K_∞ lands in $\mathbf{C}(\mathbb{R}) \subset \mathbf{G}^{\mathrm{ad}}(\mathbb{R})^+$ and $\mathbf{G}^{\mathrm{der}}(\mathbb{R})^+$ surjects onto $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})^+$, the second claim follows. □

Definition B.5. Let $(\mathbf{G}, X)$ be a Shimura–Deligne datum. We say that a compact open subgroup $K \subset \mathbf{G}(\mathbb{A}_f)$ is *sufficiently small* if, for every $g \in \mathbf{G}(\mathbb{A}_f)$, the image of the discrete subgroup $\mathbf{G}(\mathbb{Q}) \cap gKg^{-1}$ in $\mathbf{G}_{\mathrm{ad}}(\mathbb{Q})$ has no non-trivial stabilisers for its action on any connected component of X. For instance, this happens when K is *neat*, by [Pin88, Proposition 3.3(b)].

Definition B.6. Given a Shimura–Deligne datum $(\mathbf{G}, X)$, and a compact open subgroup $K \subset \mathbf{G}(\mathbb{A}_f)$, we define the *Shimura–Deligne variety* to be the double quotient

$$\mathbf{G}(\mathbb{Q})\backslash[X \times (\mathbf{G}(\mathbb{A}_f)/K)]$$

which we denote by $\mathrm{Sh}_{\mathbf{G}}(X, K)(\mathbb{C})$ or just $\mathrm{Sh}_{\mathbf{G}}(K)(\mathbb{C})$.

Arguing as in [Lan12, §2.5] (or [GN09, Lemma 4.6.1]), $\mathrm{Sh}_{\mathbf{G}}(K)(\mathbb{C})$ is a disjoint union of quotients of Hermitian symmetric domains/finite discrete sets by arithmetic subgroups, and therefore are the complex points of a quasi-projective $\mathbb{C}$-variety $\mathrm{Sh}_{\mathbf{G}}(K)_{\mathbb{C}}$ by the theorem of Baily–Borel. By [Pin88, §0.1], these varieties are also smooth for K sufficiently small. As K gets smaller, these varieties form a projective system whose limit is a quasi-compact separated scheme carrying a continuous action of $\mathbf{G}(\mathbb{A}_f)$ [Del71, §1.8]. We shall denote this scheme by $\mathrm{Sh}_{\mathbf{G}}(X)_{\mathbb{C}}$ or just $\mathrm{Sh}_{\mathbf{G},\mathbb{C}}$.

Let $u\colon (\mathbf{G}_1, X_1) \to (\mathbf{G}_2, X_2)$ be a morphism of Shimura–Deligne datum. For compact open subgroups $K_i \subset \mathbf{G}_i(\mathbb{A}_f)$ such that $u(K_1) \subset K_2$, we have an induced $\mathbb{C}$-morphism of corresponding $\mathbb{C}$-varieties by Borel's Theorem (see [Del71, §1.14]). If one starts with a closed immersion of reductive groups, this morphism is finite unramified for sufficienlty small compatible compact opens, and factors as a composition of a closed immersion followed by a finite étale morphism [Del71, Proposition 1.15].

B.2. **Canonical models.** Let $\mathbf{G}$ be a reductive $\mathbb{Q}$-group. For any $\mathbb{Q}$-algebra R, the group $\mathbf{G}(R)$ acts on the left on the space $\mathrm{Hom}_R(\mathbb{G}_{m,R}, \mathbf{G}_R)$ of algebraic group homomorphisms over R, by conjugation on the target. Let $Y = Y_{\mathbf{G}}$ be the (fppf sheafification of) the functor

$$\mathbb{Q}\text{-algebras} \to \mathbf{Sets} \qquad R \mapsto \mathbf{G}(R)\backslash \mathrm{Hom}_R(\mathbb{G}_{m,R}, \mathbf{G}_R).$$

If $F/\mathbb{Q}$ is a finite Galois extension over which $\mathbf{G}$ splits, then restricted to F-algebras, Y is a constant functor, hence representable as $\bigsqcup \mathrm{Spec}\, F$. By Galois descent, one sees that it is representable by an étale $\mathbb{Q}$-scheme.

Suppose moreover that $(\mathbf{G}, X)$ is Shimura–Deligne datum. The cocharacters $h_{\mathbb{C}} \circ \mu\colon \mathbb{G}_{m,\mathbb{C}} \to \mathbf{G}_{\mathbb{C}}$ for $h \in X$ lie in the same $\mathbf{G}(\mathbb{C})$-conjugacy class (as h lies in a single $\mathbf{G}(\mathbb{R})$-conjugacy class). Therefore, one obtains a geometric point $\mu_X \in Y(\mathbb{C}) = Y(\overline{\mathbb{Q}}) \subset Y_{\overline{\mathbb{Q}}}$.

Definition B.7. The *reflex field* $E(\mathbf{G}, X)$ of a Shimura–Deligne datum $(\mathbf{G}, X)$ is the field of moduli of the point $\mu_X \in Y_{\mathbf{G}}(\overline{\mathbb{Q}})$.

The reflex field is also the field of definition of the conjugacy class μ_X, which is how it is usually defined and computed. We give an example below.

Example B.1. Let $\mathbf{G} = \mathrm{GU}(p, q)$ be the unitary group defined in §2.1 using the imaginary quadratic field E and let $h\colon \mathbb{S} \to \mathbf{G}_{\mathbb{R}}$ be the map $z \mapsto (z, \dots, z, \bar{z}, \dots, \bar{z})$ where there are p copies of z, q copies of $\bar{z}$. Then $(\mathbf{G}, \{h\})$ is a Shimura–Deligne datum. The cocharacter μ_h associated to h over $\mathbb{C}$ is given by $\mu_h\colon \mathbb{G}_m \to \mathbb{G}_m \times \mathrm{GL}_{n,\mathbb{C}}$ $z \mapsto (z, \mathrm{diag}(z, , \dots, z, 1, \dots 1))$ which is defined over E. If $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$ is the complex conjugation, then $\sigma(\mu_h)\colon z \mapsto (z, \mathrm{diag}(1,\dots,1,z,\dots,z))$. These are in two different conjugacy classes if and only if $p \neq q$, i.e. the reflex field is E if $p \neq q$ and $\mathbb{Q}$ otherwise.

For the notion of *models* of $\mathrm{Sh}_{\mathbf{G},\mathbb{C}}$ over subfields $E \subset \mathbb{C}$, we refer the reader to [Del71, §3]. If a model exists, we denote it by $\mathrm{Sh}_{\mathbf{G},E}$. As in [Pin88, §11.4], one can define a model of $\mathrm{Sh}_{\mathbf{G},\mathbb{C}}$ over its reflex field when $\mathbf{G}$ is a torus (and $X = \{h\}$ is a singleton). Moreover, if $(\mathbf{G}_1, X_1) \to (\mathbf{G}_2, X_2)$ is a morphism, then $E(\mathbf{G}_1, X_1) \supset E(\mathbf{G}_2, X_2)$ because one obtains a morphism $Y_{\mathbf{G}_1} \to Y_{\mathbf{G}_2}$ with μ_{X_1} mapping to μ_{X_2}. This motivates the following definition.

Definition B.8. A *canonical model* of $\mathrm{Sh}_{\mathbf{G},\mathbb{C}}$ is a model $\mathrm{Sh}_{\mathbf{G},E}$ defined over the reflex field $E = E(\mathbf{G}, X)$ such that for every Shimura–Deligne datum $(\mathbf{T}, X')$ with reflex field E', where $\mathbf{T}$ is a torus, and any injective morphism $(\mathbf{T}, X') \to (\mathbf{G}, X)$, the induced map

$$\mathrm{Sh}_{\mathbf{T},\mathbb{C}} \to \mathrm{Sh}_{\mathbf{G},\mathbb{C}}$$

is the pullback of a morphism $\mathrm{Sh}_{\mathbf{T},E'} \to \mathrm{Sh}_{\mathbf{G},E} \times_{\mathrm{Spec}\, E} \mathrm{Spec}\, E'$.

Lemma B.9. *Let $(\mathbf{G}, X)$ be a Shimura–Deligne datum and $E = E(\mathbf{G}, X)$ its reflex field. Let S be the collection of all subfields $F \subset \mathbb{C}$ that arise as follows: there exists an injective morphism of Shimura–Deligne datum $(\mathbf{T}, X') \to (\mathbf{G}, X)$ such that $F = E(\mathbf{T}, X')$. Then S is non-empty and $\bigcap_{F\in S} F = E$.*

This result is [Del71, Theorem 5.1] and is the key input of the next theorem. While (SD3) is a running hypothesis in §5 of *op.cit.*, the proof of this result does not need it. As this might not be immediately obvious, we provide an exposition of the key steps.

Proof. Let $Y_{\mathbf{G}}$ be the $\mathbb{Q}$-scheme as above and let $Y_0 \cong \mathrm{Spec}\, E$ be the finite étale $\mathbb{Q}$-subscheme whose $\overline{\mathbb{Q}}$-points consist of the Galois translates of $\mu_X \in Y_{\mathbf{G}}(\overline{\mathbb{Q}})$. Let V be the $\mathbb{Q}$-scheme of regular elements in $\mathrm{Lie}(\mathbf{G})$, i.e. points in $V(R)$ are elements $v_R \in \mathrm{Lie}(\mathbf{G})_R$ whose centraliser is a maximal torus $\mathbf{T}_{v_R}$ in $\mathbf{G}_R$. Let W be (fppf sheafification of) the functor that associates to each $\mathbb{Q}$-algebra R the set of all triplets $(\mathbf{T}_R, v_R, \lambda_R)$ where

- $\mathbf{T}_R$ is a (fibrewise) maximal torus in $\mathbf{G}_R$.
- $v_R \in \mathrm{Lie}(\mathbf{T})_R$ is a regular element of $\mathrm{Lie}(\mathbf{G})_R$, i.e. the centraliser of v in $\mathbf{G}_R$ is $\mathbf{T}_R$.
- $\lambda_R\colon \mathbb{G}_{m,R} \to \mathbf{T}_R$ is a cocharacter such that the class of the induced morphism $\mathbb{G}_{m,R} \to \mathbf{G}_R$ is in $Y_{0,R}$.

Then $W_{\overline{\mathbb{Q}}} \to V_{\overline{\mathbb{Q}}}$ has constant finite fibres. By Galois descent, W is representable by a finite-type $\mathbb{Q}$-scheme admitting a finite étale surjective map $f\colon W \to V$ and a map $p\colon W \to Y_0$. This makes W a E-scheme.

$$\begin{array}{ccc} W & \xrightarrow{\ f\ } & V \\ \downarrow{\scriptstyle p} & & \downarrow \\ Y_0 & \longrightarrow & \mathrm{Spec}\,\mathbb{Q} \end{array} \tag{B.10}$$

For all $v \in V(\mathbb{R})$, $w \in W(\mathbb{C})$ with $f(w) = v$ and $\lambda_w\colon \mathbb{G}_{m,\mathbb{C}} \to \mathbf{T}_{v,\mathbb{C}}$ the cocharacter part of w, there is a unique $h_w\colon \mathbb{S} \to \mathbf{T}_{\mathbb{R}}$ such that $h_{w,\mathbb{C}} \circ \mu = \lambda_w$; on $\mathbb{R}$-points, it is given by

$$h_w(z) = \lambda_w(z) \cdot \overline{\lambda_w(z)}, \qquad z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times.$$

We note that if $v \in V(\mathbb{Q})$ is such that

- the $\mathbb{Q}$-scheme $Z_v := f^{-1}(v)$ is the spectrum of a field E_v,
- for some $w \in Z_v(\mathbb{C})$, $h_w \in X$.

then, $E_v \in S$. Indeed, the second point means that there is an injective morphism $(\mathbf{T}_v, \{h_w\}) \hookrightarrow (\mathbf{G}, X)$ of Shimura–Deligne datum, and the first guarantees that the cocharacter $\lambda_w = \mu \circ h_{w,\mathbb{C}}$ associated to the datum $(\mathbf{T}_v, \{h\})$ has field of definition E_v. The goal therefore is to show that there are many such v.

Step 1. W is geometrically irreducible as an E-scheme.

Let W' be the geometric fibre of p over μ_X and C the scheme over $\overline{\mathbb{Q}}$ of cocharacters of $\mathbf{G}_{\overline{\mathbb{Q}}}$ whose conjugacy class is μ_X. Let $q\colon W' \to C$ be the natural map. Then C, being a quotient of $\mathbf{G}_{\overline{\mathbb{Q}}}$, is irreducible and the fibers of q are isomorphic. Let $W'' \subset W'$ be the fibre of q over a given $\lambda\colon \mathbb{G}_m \to \mathbf{G}_{\overline{\mathbb{Q}}} \in C(\overline{\mathbb{Q}})$ and D the scheme of all maximal tori of $\mathbf{G}_{\overline{\mathbb{Q}}}$ that contain the image of λ. Let $r\colon W'' \to D$ be the natural map. Any maximal torus coming from D centralizes $\lambda(\mathbb{G}_m)$ and therefore must be contained in the centralizer $Z_{\mathbf{G}_{\overline{\mathbb{Q}}}}(\lambda(\mathbb{G}_m))$. Now, $Z_{\mathbf{G}_{\overline{\mathbb{Q}}}}(\lambda(\mathbb{G}_m))$ is connected by [Mil17, Theorem 17.38] and D is a quotient of $Z_{G_{\overline{\mathbb{Q}}}}(\lambda(\mathbb{G}_m))$ by [Mil17, Theorem 17.10]. Therefore, D irreducible. As $W'' \to D$ is open in the vector bundle over D associated to the Lie algebra of the universal torus over D, it has geometrically irreducible fibers over D, and is therefore irreducible too. Collecting these statements together, we see that the fibers of q are irreducible, whence W' is irreducible. As the geometric fibres of p are isomorphic and Y_0 is irreducible, the claim follows.

Step 2. In the analytic topology of $V(\mathbb{R})$, there is a non-empty open subset U such that for all $v \in U$, $\mathbf{T}_v$ contains the image of some $h \in X$.

We let $U \subset V(\mathbb{R})$ denote the subset of all $v \in V(\mathbb{R})$ such that $(\mathbf{T}_v/\mathbf{Z}_\mathbb{R})(\mathbb{R})$ is compact. Then U is open since the association $v \mapsto \mathbf{T}_v/\mathbf{Z}_\mathbb{R}$ is continuous and the set of anisotropic (equivalently compact) tori is an open subspace[6]. of the moduli space of all maximal tori of $\mathbf{G}^{\mathrm{ad}}_\mathbb{R}$. Moreover, U is non-empty as follows: for any $h \in X$, the map $h^{\mathrm{ad}} : \mathbb{S} \to \mathbf{G}^{\mathrm{ad}}_\mathbb{R}$ factors via $h^{\mathrm{ad}} : \mathrm{U}_1 \to \mathbf{G}^{\mathrm{ad}}_\mathbb{R}$. Hence any maximal torus of $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})$ containing the image of h^{ad} is contained in the centralizer $\mathrm{Stab}_\mathbf{G}(h^{\mathrm{ad}})$, and is therefore compact. Any regular element in the Lie algebra of the inverse image in $\mathbf{G}$ of $\mathbf{T}$ is then the desired element. It now remains to show that for *all* $v \in U$, $\mathbf{T}_v$ contains the image of some $h \in X$. To this end, fix $h_0 \in X$ and let K_∞ be the stabiliser of h_0 in $\mathbf{G}(\mathbb{R})$. Then $K'_\infty = K_\infty \cap \mathbf{G}^{\mathrm{der}}(\mathbb{R})^+$ is a maximal compact Lie subgroup of $\mathbf{G}^{\mathrm{der}}(\mathbb{R})^+$, and $K_\infty/\mathbf{Z}(\mathbb{R})$ the maximal compact subgroup of $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})^+$ by Lemma B.4. As any two maximal compact subgroups in $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})^+$ are conjugate and $\mathbf{T}_v/\mathbf{Z}_\mathbb{R}(\mathbb{R})$ is compact, there exists $g \in \mathbf{G}^{\mathrm{der}}(\mathbb{R})^+$ such that $g\mathbf{T}_v(\mathbb{R})g^{-1} \subseteq K_\infty$. Since $g\mathbf{T}_v(\mathbb{R})g^{-1}$ is a maximal torus in K_∞, it contains the centre of K_∞, and in particular the image of h_0. Therefore, $\mathbf{T}_v$ contains the image of $g^{-1}h_0g$.

Step 3. For any finite extension E' of E, there exists $v \in V(\mathbb{Q}) \cap U$ such that $f^{-1}(v)$ is the spectrum of a field linearly disjoint from E'.

This is (a slighly general form) of Hilbert's irreducibilty theorem applied to the diagram in (B.10), and is proved in [Del71, Lemma 5.1.3]. ☐

Remark B.11. In fact, the proof shows that there are two extensions in S whose intersection is E.

Theorem B.12 (Functoriality)**.** *Let $(\mathbf{G}_1, X_1) \to (\mathbf{G}_2, X_2)$ be a morphism of Shimura–Deligne datum and, for $i = 1, 2$, let E_i be the reflex field of $(\mathbf{G}_i, X_i)$. Suppose $\mathrm{Sh}_{\mathbf{G}_i,E_i}$ is a canonical model over E_i of $\mathrm{Sh}_{\mathbf{G}_i,\mathbb{C}}$. Then the morphism*

$$\mathrm{Sh}_{\mathbf{G}_1,\mathbb{C}} \to \mathrm{Sh}_{\mathbf{G}_2,\mathbb{C}}$$

is the pullback of a morphism $\mathrm{Sh}_{\mathbf{G}_1,E_1} \to \mathrm{Sh}_{\mathbf{G}_2,E_2} \times_{\mathrm{Spec}\, E_2} \mathrm{Spec}\, E_1$.

Proof. This is [Del71, Corollary 5.4] whose proof relies on Proposition 5.2, 5.3, and Theorem 5.1 in *op.cit.* Proposition 5.2 of *op.cit.* holds in our case as we still have real approximation, and Lemma 5.3 is a general fact about descent data of schemes (see the explanation in [Pin88, Lemma 11.8] and the reference therein). The key input of Theorem 5.1 is that the intersection of the reflex fields of tori admitting a map to $(\mathbf{G}_1, X_1)$ is E_1 (without assuming (SD3)), and this was shown in Lemma B.9 above. ☐

Corollary B.13 (Uniqueness)**.** *Canonical models are unique up to a unique isomorphism.*

Proof. If $\mathrm{Sh}_{\mathbf{G},E}$ and $\mathrm{Sh}'_{\mathbf{G},E}$ are two canonical models over the reflex field E of a datum $(\mathbf{G}, X)$, then the identity map on $\mathrm{Sh}_{\mathbf{G},\mathbb{C}}$ is the pullback of a morphism $\mathrm{Sh}'_{\mathbf{G},E} \to \mathrm{Sh}_{\mathbf{G},E}$. ☐

Corollary B.14 (Existence criterion)**.** *Let $(\mathbf{G}, X) \to (\mathbf{G}', X')$ be an injective morphism of Shimura–Deligne datum. If $(\mathbf{G}', X')$ admits a canonical model, then so does $(\mathbf{G}, X)$.*

Proof. This is [Del71, Corollary 5.7] and the proof carries over verbatim. ☐

Definition B.15. Let $(\mathbf{G}, X)$ be a Shimura–Deligne datum. In addition to the axioms introduced in Definition B.1, some additional axioms are often imposed which we list here:

[6]The space of conjugacy classes of maximal tori in $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})$ form finitely many connected components, each of which is open, and any two in the same component conjugate by $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})^+$. See the footnote in [Mil03, p. 117].

(SD3) $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})$ has no $\mathbb{Q}$-simple factors that are $\mathbb{R}$-anisotropic.
(SD4) The *weight morphism* $w_X := w \circ h \colon \mathbb{G}_{m,\mathbb{R}} \to \mathbf{G}_{\mathbb{R}}$, a priori defined over $\mathbb{R}$, is defined over $\mathbb{Q}$.
(SD5) $\mathbf{Z}_{\mathbf{G}}(\mathbb{Q})$ is discrete in $\mathbf{Z}_{\mathbf{G}}(\mathbb{A}_f)$.

Remark B.16. When $(\mathbf{G}, X)$ satisfies (SD3), we arrive at the common definition of *Shimura datum* in the literature. We note that (SD5) is equivalent to the statement that the maximal anisotropic $\mathbb{Q}$-subtorus of $\mathbf{Z}_{\mathbf{G}}$ remains anisotropic (equivalently, compact) over $\mathbb{R}$, i.e. $\mathbf{Z}_{\mathbf{G}}$ is an almost direct product of a split and a compact type torus over $\mathbb{Q}$ [Mil03, Theorem 5.36]. (SD5) implies (SD4) (see [Pin92, 5.4]) and is therefore a practical means to check the latter. Axiom (SD5) also implies that the (right) action of any sufficiently small compact open $K \subset \mathbf{G}(\mathbb{A}_f)$ on the quotient $\mathbf{G}(\mathbb{Q})\backslash[X \times \mathbf{G}(\mathbb{A}_f)]$ is free.

B.3. **PEL-type Shimura–Deligne varieties.** For the notion of (semisimple) PEL-datum, we refer the reader to [Tor19, Definition 7.2]. Here, we simply recall that a PEL datum is a tuple $(B, *, V, \langle\cdot,\cdot\rangle, h)$ where

- B is a finite-dimensional semisimple $\mathbb{Q}$-algebra
- $*$ is a positive anti-involution of B
- V is a finite-dimensional B-module
- $\langle\cdot,\cdot\rangle \colon V \times V \to \mathbb{Q}$ a non-degenerate alternating (i.e. skew Hermitian) pairing
- $h \colon \mathbb{C} \to \mathrm{End}_{B_{\mathbb{R}}} V_{\mathbb{R}}$ is an $\mathbb{R}$-algebra homomorphism

satisfying the conditions in *loc.cit.*. Given such a PEL datum as above, we can associate to it an algebraic group $\mathbf{G}$ (over $\mathbb{Q}$) whose R-points equal

$$\mathbf{G}(R) = \left\{ \mathrm{Aut}_{B_R}(V_R) \,|\, \exists \mu(g) \in R^{\times} \text{ such that } \langle gu, gv\rangle = \mu(g)\langle u, v\rangle \right\}.$$

The group $\mathbf{G}$ is reductive and $\mu \colon \mathbf{G} \to \mathbb{G}_m$ is a homomorphism, which we call the *similitude factor*. We let $\mathbf{G}_1$ denote the kernel of μ. Then $\mathbf{G}_{1,\mathbb{R}}$ splits into a product of unitary (A), symplectic (C) and orthogonal factors (D) [Tor19, Lemma 7.5 (i)]. In particular, if B is simple then only one type can occur. The $\mathbb{R}$-algebra homomorphism $h \colon \mathbb{C} \to \mathrm{End}_{B_{\mathbb{R}}}(V_{\mathbb{R}})$ gives rise to a morphism $\mathbb{S} \to \mathbf{G}$, which we still denote by h.

Lemma B.17. *If $\mathbf{G}_1$ has no factors of type (D), then the pair $(\mathbf{G}, h)$ is a Shimura–Deligne datum. In addition, it satisfies (SD4).*

Proof. This is [Tor19, Lemma 7.5 (ii)], with the only modification being that we include non-central factors of compact type. The second statement follows by Lemma 7.5 (iii) of *op.cit.* □

Definition B.18. A *PEL-type Shimura–Deligne datum* is a pair $(\mathbf{G}, h)$ that arises from a choice of a PEL datum with no orthogonal factors as above.

To any PEL-datum as above, and for any compact open subgroup $K \subset \mathbf{G}(\mathbb{A}_f)$, we have an associated moduli problem $\mathcal{M}_K$ over the reflex field which classifies abelian varieties with corresponding PEL structure as in [Lan12, §1.2]. If K is *neat*, then $\mathcal{M}_K$ is representable by a smooth quasi-projective scheme.

Remark B.19. Here, we are assuming that such a *rational* PEL-datum arises from a choice of *integral* PEL-datum as in [Lan12, §1.1]. As we are only interested in moduli problems over reflex fields in the sequel, the choice of an integral PEL datum does not matter by Corollary 1.4.3.7 of [Lan13].

Lemma B.20. *Let $(\mathbf{G}, X)$ be a PEL-type Shimura–Deligne datum arising from $(B, *, V, \langle\cdot,\cdot\rangle, h)$. Set*

$$\ker^1(\mathbb{Q}, \mathbf{G}) := \ker\left(\mathrm{H}^1(\mathbb{Q}, \mathbf{G}) \to \bigoplus_{v \leq \infty} \mathrm{H}^1(\mathbb{Q}_v, \mathbf{G}) \right).$$

Then $\ker^1(\mathbf{G}, \mathbb{Q})$ is finite and classifies isomorphism classes of skew-Hermitian B-modules $(V', \langle\cdot,\cdot\rangle')$ such that the $B_{\mathbb{A}}$-module $(V'_{\mathbb{A}}, \langle\cdot,\cdot\rangle')$ is isomorphic to $(V_{\mathbb{A}}, \langle\cdot,\cdot\rangle)$.

Proof. That $\ker^1(\mathbb{Q}, \mathbf{G})$ is finite follows by [BS64, §7]. By [Mil17, Proposition 3.50]), $\mathrm{H}^1(\mathbb{Q}, \mathbf{G})$ (resp. $\mathrm{H}^1(\mathbb{Q}_v, \mathbf{G}_v)$) classifies isomorphism classes of $\mathbf{G}$-torsors over $\mathbb{Q}$ (resp. $\mathbf{G}_v$ torsors over $\mathbb{Q}_v$), and so $\ker^1(\mathbb{Q}, \mathbf{G})$ classifies isomorphism classes of $\mathbf{G}$ torsors over $\mathbb{Q}$ that are locally trivial. Now $\mathbf{G}$-torsors are just symplectic B-modules V' that are isomorphic to V over $\overline{\mathbb{Q}}$. □

Proposition B.21. *Let* $(\mathbf{G}, X)$ *be a PEL-type Shimura–Deligne datum associated with the tuple* $(B, *, V, \langle\cdot,\cdot\rangle, h)$*, and fix an integral PEL-datum satisfying* [Lan12, Condition 1.2.5]*. For any neat compact open subgroup* $K \subset \mathbf{G}(\mathbb{A}_f)$*, one has an open and closed embedding of* $\mathbb{C}$*-varieties*

$$\mathrm{Sh}_{\mathbf{G},\mathbb{C}}(K) \hookrightarrow \mathcal{M}_{K,\mathbb{C}}. \tag{B.22}$$

If the group $\mathbf{G}$ *satisfies the Hasse principle, i.e.* $\ker^1(\mathbb{Q}, \mathbf{G}) = 0$*, then the embedding in (B.22) is an isomorphism.*

Proof. The embedding in (B.22) follows from [Lan12, Lemma 2.5.6] and the paragraph following its proof. More precisely, it is shown that $\mathrm{Sh}_{\mathbf{G}}(K)(\mathbb{C})$ is isomorphic to the locus in $\mathcal{M}_K(\mathbb{C})$ parameterising all PEL abelian varieties whose first (rational) homology is isomorphic to V (as polarised symplectic spaces with B-structure). Note that for any point in $\mathcal{M}_K(\mathbb{C})$, the first homology (over $\mathbb{A}$) of the associated abelian variety is always isomorphic to $V \otimes_{\mathbb{Q}} \mathbb{A}$ (as polarised symplectic spaces with B-structure), by definition; so to show (B.22) is an isomorphism, it is enough to show that there is a unique polarised symplectic B-module W such that $W \otimes_{\mathbb{Q}} \mathbb{A} \cong V \otimes_{\mathbb{Q}} \mathbb{A}$. This follows by Lemma B.20. ☐

Remark B.23. The discussion above can also be found in [Kot92, §8] under the assumption that B is simple.

Remark B.24. The computation of $\ker^1(\mathbb{Q}, \mathbf{G})$ can be reduced to $\ker^1(\mathbb{Q}, \mathbf{G}/\mathbf{G}^{\mathrm{der}})$ by [Mil17, Proposition 25.71], as $\mathbf{G}^{\mathrm{der}}$ is always simply connected for PEL-type Shimura Deligne datum ([Mil03, Proposition 8.7]).

Corollary B.25. *Keeping the same notation as in Proposition B.21, suppose that* $\mathbf{G}$ *satisfies the Hasse principle. Then* $\mathrm{Sh}_{\mathbf{G},\mathbb{C}}(K)$ *has a canonical model isomorphic to* $\mathcal{M}_{K,E}$*, where* $E = E(\mathbf{G}, X)$ *is the reflex field.*

Proof. Any PEL-type Shimura–Deligne datum has an embedding into the standard Siegel Shimura datum, so $\mathrm{Sh}_{\mathbf{G},\mathbb{C}}(K)$ has a unique canonical model by Corollaries B.14 and B.13. The fact that $\mathcal{M}_{K,E}$ is a canonical model follows from [Mil03, Proposition 14.12] with the identity map (the proof of this proposition is purely a statement about CM abelian varieties). ☐

Example B.2. Let $(\mathscr{G}, \mathfrak{X}) \in \{(\mathbf{H}, X_{\mathbf{H}}), (\mathbf{G}, X_{\mathbf{G}})\}$ be the Shimura–Deligne datum introduced in section 2.2. Then $(\mathscr{G}, \mathfrak{X})$ is of PEL-type and satisfies (SD5) and [Lan12, Condition 1.2.5]. Indeed

- If $\mathscr{G} = \mathbf{G}$, then the semisimple algebra is just $B = E$ and V is the $2n$-dimensional Hermitian space with Hermitian form corresponding to the matrix $J_{1,2n-1}$ (see section 2.1). In particular, this arises from an integral PEL-datum with semisimple algebra $\mathcal{O}_E$.
- If $\mathscr{G} = \mathbf{H}$, then we take $B = E \times E$ and consider the product of Hermitian spaces $V = W_1 \oplus W_2$ where W_1 (resp. W_2) has Hermitian form given by the matrix $J_{1,n-1}$ (resp. $J_{0,n}$). As above, this arises from an integral PEL-datum with semisimple algebra $\mathcal{O}_E \times \mathcal{O}_E$.

In addition to this, the group $\mathbf{G}$ satisfies the Hasse principle, for the following reason. By Remark B.24, it is enough to deduce it for the the maximal abelian quotient of $\mathscr{G}$, denoted $\mathbf{T}_{\mathscr{G}}$. In either case, $\mathbf{T}_{\mathscr{G}}$ is a product of factors U(1), GU(1) or $\mathbb{G}_m$. The Hasse principle is known to hold for these three groups; for $U(1)$, see ([Kot92, §7]), while for GU(1), $\mathbb{G}_m$, it follows by Hilbert's Theorem 90.

Appendix C. Ancona's construction for Shimura–Deligne varieties

In this appendix, we document certain results, particularly the functoriality of motivic lifts, of [Anc15] and [Tor19] that hold in the absence of (SD3). The techniques used by these authors involve mixed Shimura varieties, which are a generalisation of the usual (pure) Shimura varieties. The definition of a mixed Shimura datum includes a counterpart of axiom (SD3); axiom (vii) of [Pin88, Definition 2.1]), and as in the case of pure Shimura data, this axiom is usually invoked when strong approximation is needed (for example, when describing the connected components of mixed Shimura varieties in [Pin88, Proposition 3.9]). It also plays a key role in reducing many statements to ones involving pure Shimura data. By replacing this condition with an alternative assumption (Assumption C.3) however, many proofs in [Pin88] carry over verbatim (building on the results in Appendix B).

We have not attempted to be thorough here – instead, we content ourselves with a summary of the general results we need from [Pin88], and only use them in the restricted setting of §C.3 that suffices for the purposes of this article. We continue with the notation of Appendix B until §C.5, after which we specialise to the particular groups defined in §2.1.

C.1. **Mixed Shimura–Deligne data.** Consider the following collection of data

- $\mathbf{P}$ a connected linear algebraic group over $\mathbb{Q}$
- $\mathbf{W}$ the unipotent radical of $\mathbf{P}$
- $\mathbf{U}$ a subgroup of $\mathbf{W}$ over $\mathbb{Q}$ that is normal in $\mathbf{P}$.
- $\mathcal{X}$ a left homogenous space under the group $\mathbf{P}(\mathbb{R}) \cdot \mathbf{U}(\mathbb{C}) \subset \mathbf{P}(\mathbb{C})$,
- $h\colon \mathcal{X} \to \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, \mathbf{P}_{\mathbb{C}})$ a $\mathbf{P}(\mathbb{R}) \cdot \mathbf{U}(\mathbb{C})$-equivariant map.

Set $\mathbf{V} = \mathbf{W}/\mathbf{U}$, $\mathbf{G} = \mathbf{P}/\mathbf{W}$ and $\pi\colon \mathbf{P} \to \mathbf{G}$, $\pi'\colon \mathbf{P} \to \mathbf{P}/\mathbf{U}$ the natural projections. Then such a collection is said to be *mixed Shimura–Deligne datum* if it satisfies axioms (i)–(viii) of [Pin88, Definition 2.1], except possibly (vii). If these axioms are satisfied, the data is determined by the triple $(\mathbf{P}, \mathcal{X}, h)$ since $\mathbf{U}$ is characterised by its action on $\mathrm{Lie}(\mathbf{P})$. We suppress the dependency on h when a choice has been made, and denote such a datum by $(\mathbf{P}, \mathcal{X})$ only.

A *morphism* of mixed Shimura–Deligne datum $(\mathbf{P}_1, \mathcal{X}_1, h_1) \to (\mathbf{P}_2, \mathcal{X}_2, h_2)$ is a morphism $\phi\colon \mathbf{P}_1 \to \mathbf{P}_2$ and a $\mathbf{P}_1(\mathbb{R}) \cdot \mathbf{U}_1(\mathbb{C})$-equivariant map $\Psi\colon \mathcal{X}_1 \to \mathcal{X}_2$ satisfying the commutativity property of [Pin88, Definition 2.3]. We call a morphism *injective* if both ϕ and Ψ are.

Given a mixed Shimura–Deligne datum $(\mathbf{P}, \mathcal{X}, h)$ and a compact open subgroup $K \subset \mathbf{P}(\mathbb{A}_f)$, we define the corresponding *mixed Shimura–Deligne variety* to be the double coset

$$\mathrm{Sh}_{\mathbf{P}}(\mathcal{X}, K)(\mathbb{C}) := \mathbf{P}(\mathbb{Q}) \backslash \left[\mathcal{X} \times \mathbf{P}(\mathbb{A}_f)/K\right].$$

As in [Pin88, Proposition 2.19], every connected component of $\mathcal{X}$ is a holomorphic complex vector bundle on a Hermitian symmetric domain, or simply a complex vector space. Therefore, if K is neat, then $\mathrm{Sh}_{\mathbf{P}}(\mathcal{X}, K)(\mathbb{C})$ inherits the structure of a complex analytic variety. A morphism of mixed Shimura–Deligne data induces a morphism on the corresponding mixed Shimura–Deligne varieties when the compact open subgroups are chosen as in [Pin88, Proposition 3.8].

C.2. **Hodge structures.** Let $(\mathbf{G}, X)$ be a Shimura–Deligne datum as in Definition B.1, and let F be a number field. Let $\mathrm{Rep}_F(\mathbf{G})$ denote the category of all finite-dimensional algebraic representations $(\rho, \mathbf{V})$ of $\mathbf{G}_F$ and set $V = \mathbf{V}(F)$, the underlying F-vector space. Such a representation can equivalently be viewed as an algebraic representation of $\mathbf{G}$ defined over $\mathbb{Q}$ of dimension $\dim_F(V) \cdot [F : \mathbb{Q}]$, together with a $\mathbb{Q}$-algebra homomorphism $F \hookrightarrow \mathrm{End}_{\mathbf{G}(\mathbb{Q})}(V)$.

For any $(\rho, \mathbf{V}) \in \mathrm{Rep}_F(\mathbf{G})$ and $h \in X$, we obtain a Hodge structure on $V \otimes_{\mathbb{Q}} \mathbb{C}$ via the map $\rho \circ h$. In particular, $V \otimes_{\mathbb{Q}} \mathbb{C}$ breaks up into one-dimensional subspaces for which $\mathbb{C}^\times \subset \mathbb{S}_{\mathbb{C}}$ acts via a character of the form $z \mapsto z^{-p_i} \bar{z}^{-q_i}$ – the corresponding Hodge bigrading of this subspace is given by the pair (p_i, q_i), and its weight is defined to be $p_i + q_i$. The *Hodge type* of $(\rho, \mathbf{V})$ is the collection of all such pairs that appear in this decomposition. If $p_i + q_i = n$ for all i, then we say that $\mathbf{V}$ is pure of weight n. In this case, for any $h \in X$, the weight homomorphism $\rho \circ h \circ w\colon \mathbb{G}_{m,\mathbb{R}} \to \mathrm{GL}(V \otimes_{\mathbb{Q}} \mathbb{R})$ is equal to the character $\lambda \to \lambda^n$ (recall our conventions on w at the start of Appendix B).

Definition C.1. We let $\mathrm{Rep}_F(\mathbf{G})^{\mathrm{AV}}$ denote the full sub-category of $\mathrm{Rep}_F(\mathbf{G})$ whose objects have Hodge type contained in $\{(-1, 0), (0, -1)\}$.

C.3. **Unipotent extensions.** We shall primarily be concerned with a specific sub-class of mixed Shimura–Deligne datum, namely those which are unipotent extensions and of a prescribed Hodge type.

Definition C.2. Let $(\mathbf{G}, X)$ be a Shimura–Deligne datum satisfying (SD5). For a representation $\mathbf{V}$ in $\mathrm{Rep}_F(\mathbf{G})^{\mathrm{AV}}$, consider the pair $(\mathbf{P}, \mathcal{X})$, where

- $\mathbf{P} = \mathrm{Res}_{F/\mathbb{Q}}(\mathbf{V}) \rtimes \mathbf{G}$ is the unipotent extension of $\mathbf{G}$, with natural map $\pi\colon \mathbf{P} \to \mathbf{G}$
- $\mathcal{X}$ is the subset of all $t \in \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, \mathbf{P}_{\mathbb{C}})$ which are defined over $\mathbb{R}$ and satisfy $\pi \circ t \in X_{\mathbb{C}} = \{h_{\mathbb{C}} : h \in X\}$.

Then the pair $(\mathbf{P}, \mathcal{X})$ satisfies axioms (i)–(viii) in [Pin88, 2.1], excluding axiom (vii) (which is axiom (SD3)) and therefore determines a mixed Shimura–Deligne datum. We refer to such a datum as a *unipotent extension* of the pure datum $(\mathbf{G}, X)$.

Under an additional assumption below, we will explain how one can associate mixed Shimura–Deligne varieties to the unipotent extensions in Definition C.2 which satisfy good functoriality properties in both $(\mathbf{G}, X)$ and $\mathbf{V}$. Essentially, all the results in [Pin88] that we will need hold by replacing (SD3) with the following assumption. Let $\mathbb{S}^1 \subset \mathbb{S}$ denote the circle group.

Assumption C.3 (c.f. [Pin88, 1.12])**.** *Let $(\mathbf{G}, X)$ be a Shimura–Deligne datum satisfying (SD5) and let $(\mathbf{P}, \mathcal{X})$ be a unipotent extension as in Definition C.2. Let $\mathbf{G}_1$ be a normal subgroup of $\mathbf{G}$ which contains the image of $h(\mathbb{S}^1)$, for any $h \in X$. Let $(\rho, \mathbf{M})$ be an algebraic representation of $\mathbf{P}$ (over $\mathbb{Q}$) that is pure of weight n, i.e. for any $h \in \mathcal{X}$, the action of $\rho \circ h \circ w(z)$ on $\mathbf{M}_{\mathbb{C}}$ is given by multiplication by z^n. We assume that there exists*

- *A* one-dimensional *algebraic representation $\mathbf{N}$ of $\mathbf{P}$, defined over $\mathbb{Q}$, which factors through $\mathbf{G}/\mathbf{G}_1$ and is pure of Hodge type (n, n).*
- *A $\mathbf{P}$-equivariant non-degenerate pairing $\Psi\colon \mathbf{M} \otimes_{\mathbb{Q}} \mathbf{M} \to \mathbf{N}$, and*
- *For every $h \in \mathcal{X}$, a morphism of rational Hodge structures $\lambda_h\colon \mathbf{N} \to \mathbb{Q}(-n)$, such that $\lambda_h \circ \Psi$ is a polarisation for the Hodge structure on $\mathbf{M}$ defined by h.*

Remark C.4. Assumption C.3 is an assumption on $(\mathbf{P}, \mathcal{X}, \rho, \mathbf{M})$. By abuse of language, we will say that Assumption C.3 holds for $(\mathbf{P}, \mathcal{X})$ if it holds for $(\mathbf{P}, \mathcal{X}, \rho, \mathbf{M})$, for every pure algebraic representation $(\rho, \mathbf{M})$ of $\mathbf{P}$.

Remark C.5. If $(\mathbf{G}, X)$ is a Shimura–Deligne datum satisfying (SD3) and (SD5) then Assumption C.3 holds for irreducible $\mathbf{M}$. If, in addition to this, $\dim(\mathbf{G}/\mathbf{G}_1) \leq 1$, then Assumption C.3 holds for all $\mathbf{M}$ (see [Pin88, 1.13]).

In the absence of (SD3), Assumption C.3 provides an alternative way to deduce [Tor19, Lemma 5.5]. In particular, it holds for Shimura–Deligne data of PEL-type satisfying (SD5).

Lemma C.6. *Let $(\mathbf{G}, X)$ be a PEL-type Shimura–Deligne datum as in Definition B.18 that satisfies (SD5). Then Assumption C.3 holds for any unipotent extension of $(\mathbf{G}, X)$ as in Definition C.2. In fact, one can take $\mathbf{N}$ to be a power of the similitude character.*

Proof. Let $(\mathbf{P}, \mathcal{X})$ be any unipotent extension, as in Definition C.2, and take any $t \in \mathcal{X}$ and $h \in X$ such that $\pi \circ t = h_{\mathbb{C}}$. Take $\mathbf{G}_1$ to be the kernel of the similitude character. Then this is a normal subgroup of $\mathbf{G}$ which contains $h(\mathbb{S}^1)$, so satisfies the conditions in Assumption C.3. Furthermore, we have $\mathbf{G}/\mathbf{G}_1 \cong \mathbb{G}_m$.

The homomorphism $t \circ w\colon \mathbb{G}_{m,\mathbb{C}} \to \mathbf{P}_{\mathbb{C}}$ defines a weight filtration $W_\bullet$ on the category $\mathrm{Rep}_{\mathbb{C}}(\mathbf{P})$ as in the end of §1 in [Mil90] (note that we are using ascending, rather than descending, filtrations – see Remark 1.8 in *loc.cit.*). By the assumptions on $(\mathbf{P}, \mathcal{X})$, the algebraic group $\mathbf{P}_{\mathbb{C}}$ stabilises the filtration $W_\bullet$ (see [Pin88, Proposition 1.4]). By [Mil90, Proposition 1.7], this implies that $W_0\mathbf{P}_{\mathbb{C}} = \mathbf{P}_{\mathbb{C}}$ and $W_{-1}\mathbf{P}_{\mathbb{C}} = \mathrm{Res}_{F/\mathbb{Q}}(\mathbf{V})_{\mathbb{C}}$. As a consequence, we see that any pure algebraic representation of $\mathbf{P}$ must factor through $\mathbf{G}$.

Combining this with the fact that the pairing Ψ is required to be $\mathbf{P}$-equivariant, it is enough to verify that Assumption C.3 holds in the case $(\mathbf{P}, \mathcal{X}) = (\mathbf{G}, X)$, which we place ourselves in for the remainder of this proof. It is also enough to show that for any irreducible representation $\mathbf{M}$ that is pure of weight n, the assumption is satisfied with $\mathbf{N}$ equal to the n-th power of the similitude character. Indeed, the category $\mathrm{Rep}_{\mathbb{Q}}(\mathbf{G})$ is semisimple, so we can easily extend the result to arbitrary pure representations after showing this.

Let $\mathbf{M}$ be an irreducible algebraic representation of $\mathbf{G}$. Any such representation is still irreducible after restricting to $\mathbf{G}_1$, and since the category $\mathrm{Rep}_{\mathbb{Q}}(\mathbf{G}_1)$ is semisimple, we must have

$$\dim_{\mathbb{Q}}\left((\mathbf{M} \otimes_{\mathbb{Q}} \mathbf{M})_{\mathbf{G}_1}\right) \leq 1$$

where $(-)_{\mathbf{G}_1}$ denotes coinvariants by $\mathbf{G}_1$. Furthermore this dimension is equal to 1 precisely when $\mathbf{M}|_{\mathbf{G}_1} \cong \mathbf{M}^*|_{\mathbf{G}_1}$, by Schur's lemma. Another way of saying this is that there exists an integer r such that

$$\mathbf{M}^* \cong \mathbf{M} \otimes \mu^r$$

where μ is the similitude character of $\mathbf{G}$. But the character μ gives rise to a Hodge structure that is pure of weight $-2r$, so the above isomorphism implies that $r = n$. The rest of the lemma now follows from the proof of [Pin88, 1.12] (using the fact that conjugation by $h(i)$ is a Cartan involution for $\mathbf{G}_{1,\mathbb{R}}$). ☐

C.4. **Summary of results on mixed Shimura–Deligne varieties.** Let $(\mathbf{G}, X)$ be a Shimura–Deligne datum that satisfies (SD5) and Assumption C.3 for any unipotent extension $(\mathbf{P}, \mathcal{X})$ as in Definition C.2 (see Remark C.4). This assumption will allow us to apply the majority of the results in [Pin88, §1–§3, §11] and [Tor19], which we summarise below:

(1) Let $\mathbf{V}$ be an algebraic representation of $\mathbf{G}_F$ in $\mathrm{Rep}_F(\mathbf{G})^{\mathrm{AV}}$ which we view as an algebraic representation of $\mathbf{G}$ with an F-structure, and let $(\mathbf{P} = \mathrm{Res}_{F/\mathbb{Q}}(\mathbf{V}) \rtimes \mathbf{G}, \mathcal{X})$ denote the unipotent extension as in Definition C.2. For a neat compact open subgroup $K' \subset \mathbf{P}(\mathbb{A}_f)$, we set

$$\mathrm{Sh}_{\mathbf{P}}(K')(\mathbb{C}) := \mathbf{P}(\mathbb{Q})\backslash [\mathcal{X} \times \mathbf{P}(\mathbb{A}_f)/K']$$

which we view with the quotient topology induced from $\mathcal{X}$. By [Pin88, 3.2–3.3] and Baily–Borel this set carries the structure of a complex algebraic variety, which we call the associated mixed Shimura–Deligne variety. Such an example of a neat compact open subgroup is as follows. Let $K \subset \mathbf{G}(\mathbb{A}_f)$ be a neat compact open subgroup, and let $L \subset \mathbf{V}(\mathbb{A}_f)$ denote a (full rank) $\widehat{\mathbb{Z}}$-lattice that is stable under K. By [Tor19, Lemma 5.4], $L \rtimes K$ is a neat compact open subgroup of $\mathbf{P} := \mathrm{Res}_{F/\mathbb{Q}}(\mathbf{V}) \rtimes \mathbf{G}$.

(2) Let $K', L' \subset \mathbf{P}(\mathbb{A}_f)$ be neat compact open subgroups satisfying $\sigma^{-1}L'\sigma \subset K'$ for some $\sigma \in \mathbf{P}(\mathbb{A}_f)$. Then we have a finite étale map

$$\mathrm{Sh}_{\mathbf{P}}(L')(\mathbb{C}) \xrightarrow{[\sigma]} \mathrm{Sh}_{\mathbf{P}}(K')(\mathbb{C})$$

constructed in exactly the same way as for (pure) Shimura–Deligne varieties (see [Pin88, 3.4]). We can also define morphisms of mixed Shimura–Deligne datum in the usual way, and this defines morphisms between the associated Shimura–Deligne varieties (see [Pin88, 3.8]).

(3) For $\mathbf{V}$ in $\mathrm{Rep}_F(\mathbf{G})^{\mathrm{AV}}$ the mixed Shimura–Deligne variety $\mathrm{Sh}_{\mathbf{P}}(L \rtimes K)(\mathbb{C})$ has the structure of an abelian scheme over $\mathrm{Sh}_{\mathbf{G}}(K)(\mathbb{C})$ (c.f. [Pin88, 3.22] – we use Assumption C.3 here). In particular, these abelian schemes satisfy functoriality properties with respect to the datum $(\mathbf{G}, X)$ and the representation $\mathbf{V}$ (see [Tor19, Lemma 5.7]).

(4) There is a natural way to define reflex fields and canonical models associated to mixed Shimura–Deligne data (see [Pin88, §11]). In our setting, the reflex field E of any unipotent extension $(\mathbf{P}, \mathcal{X})$ as in Definition C.2 is equal to the reflex field of $(\mathbf{G}, X)$ (11.2 in *op.cit.*). Suppose that $\mathrm{Sh}_{\mathbf{G}}(K)$ admits a canonical model over E (which is unique by Corollary B.13), then the results of §11 and the reduction lemma (Lemma 2.26) in *op.cit.* imply that $\mathrm{Sh}_{\mathbf{P}}(K')$ admits a unique canonical model over E.

Furthermore, the morphisms discussed in (2) descend to morphisms between the canonical models, as well as the functoriality properties in (3) (as long as the pure Shimura–Deligne data admit canonical models). Note that we are permitted to apply the reduction lemma in *op.cit.* by Assumption C.3.

Corollary C.7. *Take $\mathscr{G} \in \{\mathbf{G}, \mathbf{H}, \mathbf{T}, \widetilde{\mathbf{G}}\}$ to be any of the groups defined in section 2.1, and X the associated symmetric space. Then $(\mathscr{G}, X)$ is a Shimura–Deligne datum which satisfies (SD5) and Assumption C.3 for any unipotent extension $(\mathscr{P}, \mathcal{X})$ of $(\mathscr{G}, X)$ as in Definition C.2 (including, of course, the datum $(\mathscr{G}, X)$ itself). In particular, (1)–(4) above hold for any such $(\mathscr{P}, \mathcal{X})$.*

Proof. The only potentially problematic group is $\mathscr{G} = \mathbf{H}$, but we have shown that Assumption C.3 is satisfied in this case (see Lemma C.6). The existence of a canonical model for $(\mathbf{H}, X_{\mathbf{H}})$ follows from Corollary B.14 (the other groups gives rise to Shimura data in the usual sense). □

C.5. **Lifts of the canonical construction.** We now put ourselves in the situation of Corollary C.7, so $(\mathscr{G}, X) \in \{(\mathbf{G}, X_{\mathbf{G}}), (\mathbf{H}, X_{\mathbf{H}}), (\mathbf{T}, X_{\mathbf{T}}), (\widetilde{\mathbf{G}}, X_{\widetilde{\mathbf{G}}})\}$. Let p be a prime that splits in $E/\mathbb{Q}$ and recall that we have fixed an embedding $E \hookrightarrow \overline{\mathbb{Q}}_p$, which distinguishes a prime $\mathfrak{P}$ of E lying above p satisfying $E_{\mathfrak{P}} \cong \mathbb{Q}_p$. By the general procedure described in section 2.5, for any neat compact open subgroup $K \subset \mathscr{G}(\mathbb{A}_f)$ we have a $\mathbb{Q}_p$-linear tensor functor

$$\mu_{\mathscr{G},K} \colon \mathrm{Rep}_{\mathbb{Q}_p}(\mathscr{G}) \to \text{Ét}\left(\mathrm{Sh}_{\mathscr{G}}(K)\right)_{\mathbb{Q}_p}$$

from the category of finite-dimensional algebraic representations of $\mathscr{G}_{\mathbb{Q}_p}$ to lisse $\mathbb{Q}_p$-sheaves on $\mathrm{Sh}_{\mathscr{G}}(K)$.

Theorem C.8. *Let $\mathscr{G} \in \{\mathbf{G}, \mathbf{H}, \mathbf{T}, \widetilde{\mathbf{G}}\}$ as above. Then the results of* [Tor19, §10]*, excluding Lemma 10.6, hold in this setting. If $\mathscr{G} \in \{\mathbf{G}, \mathbf{H}\}$, then Lemma 10.6 does hold, in addition to the results of §8–§9 of* op.cit. *where Betti cohomology is replaced with p-adic cohomology and the functoriality statements are with respect to the embedding $\mathbf{H} \hookrightarrow \mathbf{G}$ (as in section 2.1).*

Proof. We will justify that all the proofs hold in our setting, even though we have not assumed (SD3). Note that (SD5) and Assumption C.3 hold for (any unipotent extension of) the Shimura–Deligne data that we are considering, and that the morphism $\mathbf{H} \hookrightarrow \mathbf{G}$ is admissible in the sense of [Tor19, Definition 9.1]. The key references for the proofs are [Pin88], [Pin92] and [Wil97, §I–II]. We justify the use of the results in [Wil97] (the remaining results from [Pin88] and [Pin92] are justified in section C.4).

As we are working in the special case of mixed Shimura–Deligne datum in Definition C.2, as remarked at the end of page 10 in [Wil97, §II], one can check that we are in the situation of [Wil97, §I.3] (except for geometric connectedness). Also [Pin88, 3.13] is valid in our setting, so Lemma 1.6 in [Wil97, §II] holds (the fibres of the map $\mathrm{Sh}_{\mathscr{P}}(L \rtimes K)_{\overline{E}} \to \mathrm{Sh}_{\mathscr{G}}(K)_{\overline{E}}$ are "unipotent $K(\pi, 1)$s"). These observations mean that [Wil97, §II.4] is valid in our setting. Indeed, we first check that all the cited results in this section hold:

- The reference [Pin92, Proposition 3.3.3]. It is an easy check that this proof does not depend on the axiom (SD3).
- Theorem 4.3 in [Wil97, §II] (which is a consequence of Propositions 5.5.4, 5.8.2 and 5.6.1 in [Pin92]). For part (a) there is nothing to check, because any torus satisfies (SD3). For parts (b) and (c), the proofs in [Pin92, §5.6] still hold verbatim (we do not need to consider models and compactifications). The arguments involving special points are also valid in our setting, because the results in [Pin88, §11] are valid by replacing any referenced result in [Del71] with the appropriate analogue in Appendix B.
- The results in [Wil97, §I] are valid by the above discussion.

Then one can check that none of the remaining arguments in [Wil97, §II.4] rely on the axiom (SD3). We note that the opposite convention of left/right actions for the functor $\mu_{\mathscr{G}}$ is used in *loc.cit.*, however this does not affect the validity of the results.

One can now check that the arguments in [Tor19] carry over into our situation. Furthermore, Ancona's construction is valid in our setting, as the results in [Anc15] do not depend on (SD3) (in fact the majority of the paper doesn't even involve Shimura varieties). □

Remark C.9. Some of the arguments in [Tor19] do involve passing to connected components of the mixed Shimura varieties. However, this is only ever used in an abstract way; an explicit description of the connected components is not needed, which would require (SD3).

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(Graham) Department of Mathematics, South Kensington Campus, Imperial College London, London SW7 2AZ, UK

E-mail address: `a.graham17@imperial.ac.uk`

(Shah) Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA 02138, U.S.A.

E-mail address: `swshah@math.harvard.edu`