Modular forms: problem set 6

Due August 10

Exercise 1. For $N \geq 1$, show that the map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ given by reducing the matrix coefficients mod N is surjective.

***Exercise 2.**(1) For a finite field $\mathbb{F}_q$ of order q, show that

$$\#\,\mathrm{GL}_n(\mathbb{F}_q) = \prod_{k=0}^{n-1}(q^n - q^k).$$

[Hint: construct elements of $\mathrm{GL}_n(\mathbb{F}_q)$ one row at a time; these rows must be linearly independent.]

(2) Show that for $e \geq 1$,

$$\#\,\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z}) = p^{3e}\left(1 - \frac{1}{p^2}\right).$$

[Hint: induct on e using part (1).]

(3) Show that for $N \geq 1$,

$$\#\,\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)] = N^3 \prod_{p|N}\left(1 - \frac{1}{p^2}\right).$$

(4) Show that the map

$$\begin{aligned} \Gamma_1(N) &\longrightarrow \mathbb{Z}/N\mathbb{Z} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto b \pmod N \end{aligned}$$

is surjective with kernel $\Gamma(N)$.

(5) Show that the map

$$\begin{aligned} \Gamma_0(N) &\longrightarrow (\mathbb{Z}/N\mathbb{Z})^\times \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto d \pmod N \end{aligned}$$

is surjective with kernel $\Gamma_1(N)$.

(6) Show that $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N\prod_{p|N}\left(1 + \frac{1}{p}\right)$.

Exercise 3. Show that the map

$$\begin{aligned} \Gamma(N)\backslash\mathcal{H} &\longrightarrow S(N) \\ \tau &\mapsto \left(\mathbb{C}/\Lambda_\tau, \frac{\tau}{N} + \Lambda_\tau, \frac{1}{N} + \Lambda_\tau\right) \end{aligned}$$

is a bijection. Recall that $\Lambda_\tau = \tau\mathbb{Z} + \mathbb{Z}$.

***Exercise 4.** Let $\mathcal{F} = \left\{\tau \in \mathcal{H} : |\tau| \geq 1, |\operatorname{Re}(\tau)| \leq \frac{1}{2}\right\}$, and recall that we showed that for any $\tau \in \mathcal{H}$, we have $\gamma\tau \in \mathcal{F}$ for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Show that if $\tau, \tau' \in \mathcal{F}$ are distinct points and $\gamma\tau = \tau'$ for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then either

(i) $\operatorname{Re}(\tau) = \pm\frac{1}{2}$ and $\tau' = \tau \mp 1$, or

(ii) $|\tau| = 1$ and $\tau' = \frac{-1}{\tau}$.