Modular forms: problem set 4

Due July 27

***Exercise 1.** Let $\Lambda \subseteq \mathbb{C}$ be a lattice. Give an atlas on $\mathbb{C}/\Lambda$ which turns it into a Riemann surface. Verify that if $f : \mathbb{C} \to \mathbb{C}$ is a linear map with $f(\Lambda) \subseteq \Lambda'$, then f defines a holomorphic map $\mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$.

***Exercise 2.** Recall that for any holomorphic map of Riemann surfaces $f : X \to Y$ and $x \in X$, there are neighborhoods $U \subseteq X$ of x and $V \subseteq Y$ of $f(x)$ and isomorphisms $\varphi_U : U \to U' \subseteq \mathbb{C}$ and $\varphi_V : V \to V' \subseteq \mathbb{C}$ of Riemann surfaces so that

$$\varphi_V \circ f \circ \varphi_U^{-1}(z)$$

is either 0 or z^n for some $n \geq 1$.

Define $\text{ord}_x f$ to be n (resp. ∞) if $\varphi_V \circ f \circ \varphi_U^{-1}(z) = z^n$ (resp. 0) for some φ_U, φ_V as above.

(1) Show that $\text{ord}_x f$ is well-defined. That is, show that if f is locally of the form $z \mapsto z^n$ in some coordinates and $z \mapsto z^m$ in some other coordinates, than $n = m$, and that if f is locally of the form $z \mapsto 0$ in some coordinates, then it is not locally of the form $z \mapsto z^n$ for any $n \geq 1$.

(2) Show that if $f : X \to Y$ is nonconstant then $\text{ord}_x f < \infty$ for all $x \in X$.

(3) Define the *ramification locus* of $f : X \to Y$ to be $R(f) = \{x \in X : \text{ord}_x f > 1\}$. Show that if f is nonconstant then $R(f)$ is discrete. If X is compact, deduce that $R(f)$ is finite.

Exercise 3. A map $f : X \to Y$ of topological spaces is *proper* if $f^{-1}(K) \subseteq X$ is compact for any $K \subseteq Y$ compact.

(1) Show that if $f : X \to Y$ is a continuous map of compact Hausdorff spaces, then f is proper.

(2) Show that if $f : X \to Y$ is a proper map of Riemann surfaces, then f is closed, i.e. $f(E) \subseteq Y$ is closed whenever $E \subseteq X$ is.

(3) Show that any nonconstant, proper map of Riemann surfaces is surjective. This generalizes the result that any nonconstant map of compact Riemann surfaces is surjective.

Exercise 4. If $f : X \to Y$ is a proper, nonconstant map of Riemann surfaces, define for $y \in Y$

$$d(y) = \sum_{x \in f^{-1}(y)} \text{ord}_x f \in \mathbb{N}.$$

(1) Show that d is constant as a function of $y \in Y$. This constant value is called the *degree* of f, and away from the discrete set of points $f(R(f))$, it is the number of preimages of a point in Y.

(2) If X is a compact Riemann surface and g is a meromorphic function on X, then show that $\deg \text{div}(g) = 0$.