Modular forms: problem set 3

Due July 20

***Exercise 1.** Let $K/\mathbb{Q}$ be a quadratic extension. Write down $L(\rho, s)$ for all irreducible representations ρ of $\mathrm{Gal}(K/\mathbb{Q})$. Multiply these functions together – is the result familiar?

Exercise 2. Repeat exercise 1 for $K = \mathbb{Q}(\zeta_7)$. Formulate a conjecture that should hold for any abelian extension L/K.

Exercise 3. Repeat exercise 1 for K your favorite non-abelian extension. How do you need to refine your conjecture from exercise 2 to apply to non-abelian extensions?

Students who enjoy representation theory should also like the following exercise.

Exercise 4. Let L/K be Galois.

(1) If ρ, ρ' are representations of $\mathrm{Gal}(L/K)$ then show

$$L(\rho \oplus \rho', s) = L(\rho, s)L(\rho', s).$$

(2) Let $L/K'/K$ be a Galois subextension, and let ρ be a representation of $\mathrm{Gal}(L/K')$. Show that

$$L(\rho, s) = L(\mathrm{Ind}_{\mathrm{Gal}(L/K')}^{\mathrm{Gal}(L/K)} \rho, s)$$

where $\mathrm{Ind}_{\mathrm{Gal}(L/K')}^{\mathrm{Gal}(L/K)} \rho$ is the induced representation.

(3) Prove that that

$$\zeta_L(s) = \zeta_K(s) \prod_{\text{irrep } \rho \neq 1} L(\rho, s)^{\dim \rho}$$

where the product runs over the nontrivial irreducible representations of $\mathrm{Gal}(L/K)$.

***Exercise 5.** A *Riemann surface* is a 1-dimensional complex manifold. More formally, a Riemann surface is a connected Hausdorff space X together with a collection $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ of open sets $U_\alpha \subseteq X$ and homeomorphisms $\varphi_\alpha : U_\alpha \to V_\alpha$ of U_α with an open subset $V_\alpha \subseteq \mathbb{C}$, such that $\{U_\alpha\}$ covers X and the transition maps

$$\varphi_\beta \circ \varphi_\alpha^{-1} : V_\alpha \to V_\beta$$

are holomorphic. The collection $\{(U_\alpha, \varphi_\alpha)\}$ is called an *atlas*, and we call each $(U_\alpha, \varphi_\alpha)$ a *coordinate neighborhood*. For example, $X = \mathbb{C}$ is a Riemann surface with atlas $\{(\mathbb{C}, \mathrm{id}_\mathbb{C})\}$.

(1) Let $X = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. Recall that a base of opens at $\infty \in X$ is given by the sets $\{z \in \mathbb{C} : |z| > r\} \cup \infty$ for $r > 0$. Show that $\{(\mathbb{C}, \mathrm{id}_\mathbb{C}), (X \setminus \{0\}, \varphi)\}$ where

$$\varphi(z) = \begin{cases} \frac{1}{z}, & z \in \mathbb{C} \\ 0, & z = \infty \end{cases}$$

is an atlas that turns X into a compact Riemann surface.

If X and Y are Riemann surfaces, a map $f : X \to Y$ is a function so that for every $x \in X$, there are coordinate neighborhoods (U, φ_U) of x and (V, φ_V) of $f(x) \in Y$ such that

$$\varphi_V \circ f \circ \varphi_U^{-1}|_{\varphi_U(W)} : \varphi_U(W) \to \mathbb{C}$$

is holomorphic for some $W \subseteq U$ such that $f(W) \subseteq V$. In other words, a map $f : X \to Y$ is a function that is "locally holomorphic" with respect to the given atlas.

(2) Show that a meromorphic function on $\mathbb{C}$ is the same thing as a map of Riemann surfaces $\mathbb{C} \to \mathbb{P}^1$.

(3) Show that a meromorphic function on $\mathbb{C}$ extends to a map of Riemann surfaces $\mathbb{P}^1 \to \mathbb{P}^1$.

(4) The Open Mapping Theorem says that any nonconstant map of Riemann surfaces $f : X \to Y$ is open, i.e. for any $U \subseteq X$, we have that $f(U) \subseteq Y$ is open. Using this, show that any map $f : X \to Y$ of compact Riemann surfaces is either constant or surjective. Deduce Louiville's Theorem, which says that any holomorphic function on $\mathbb{C}$ is either constant or unbounded.