Modular forms: problem set 1

Due July 6

***Exercise 1.**

(1) If $(\{A_i\}, \{\varphi_{ij}\})$ is an inverse system of groups, check that $\varprojlim_i A_i$ is a group. Do the same for an inverse system of rings.

(2) If $(\{A_i\}, \{\varphi_{ij}\})$ is an inverse system of Hausdorff topological spaces, check that the conditions $\varphi_{ij}(a_i) = a_j$ all define closed subsets of $\prod A_i$. Conclude that $\varprojlim A_i$ is a closed subset of $\prod A_i$.

Exercise 2. Let I be a partially ordered set. We call a subset $I' \subseteq I$ *cofinal* if any $i \in I$ is bounded by some $i' \in I'$, i.e. for some $i' \in I'$ we have $i \leq i'$. Show that if $I' \subseteq I$ is cofinal then the inverse systems $(\{A_i\}_{i \in I}, \{\varphi_{ij}\}_{j \leq i \in I})$ and $(\{A_i\}_{i \in I'}, \{\varphi_{ij}\}_{j \leq i \in I'})$ have isomorphic inverse limits.

***Exercise 3.** Consider the inverse system $\{\mathbb{Z}/p^n\mathbb{Z}\}_{n \geq 1}$ where the transition maps $\mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z}$ are the mod p^m reduction maps for $m \leq n$. Show that the map

$$\mathbb{Z}_p \longrightarrow \varprojlim_i \mathbb{Z}/p^i\mathbb{Z}$$
$$a \mapsto (a \mod p^i)_{i \in \mathbb{N}}$$

is an isomorphism of rings and a homeomorphism of topological spaces.

Exercise 4. Show that $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \cong \hat{\mathbb{Z}}$ where

$$\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$$

and the product is taken over the prime numbers p.

Exercise 5. A topological group G is a topological space equipped with continuous functions

$$m \colon G \times G \longrightarrow G$$
$$\iota \colon G \longrightarrow G$$

satisfying the usual group axioms for multiplication and inversion. So, for instance, there exists a point $e \in G$ (the identity) such that $m(g, \iota(g)) = m(\iota(g), g) = e$ for all $g \in G$. As an example, $\mathbb{R}$ is a topological group under $(x, y) \mapsto x + y$ and $x \mapsto -x$ with identity $0 \in \mathbb{R}$.

Let G be a topological group.

(1) Let $g \in G$. Show that the map $G \to G$ given by translation by g (i.e. $x \mapsto gx$) is a homeomorphism.

(2) Show that if $H \leq G$ is an open subgroup, then H is closed as well.

(3) If $H \leq G$ is a normal subgroup, show that G/H is a topological group under the quotient topology.

(4) If $H \leq G$ is a normal subgroup, show that G/H is Hausdorff if and only if H is closed.

(5) Show that the connected component G^0 of the identity $e \in G$ is a closed normal subgroup of G.

Exercise 6. Let L/K be Galois.

(1) Show that $\mathrm{Gal}(L/K)$ is a topological group, i.e. show that the multiplication and inversion maps are continuous.

(2) Show that for $\sigma \in \mathrm{Gal}(L/K)$, the map $\tau \mapsto \sigma\tau$ is a homeomorphism of $\mathrm{Gal}(L/K)$ with itself.

(3) Show that $\{\mathrm{Gal}(L/M) : M/K \text{ finite Galois}\}$ is a basis of open sets at the identity. Conclude that $\mathrm{Gal}(L/K)$ is totally disconnected.