Final paper topics

Topics not included here *might* be okay, subject to Sam's approval.

1 Abelian varieties

Abelian varieties are projective varieties with a compatible group structure; elliptic curves are the 1-dimensional case. Students who choose this topic should define abelian varieties, present proofs of their basic properties, and explain how some results about elliptic curves generalize to higher-dimensional abelian varieties.

Suggested source: Milne, *Abelian varieties*; introduction and first few sections. Additional sources are likely to be useful.

2 Applications of modular forms

We have used modular forms to derive formulas relating $\sigma_k(n)$ for different values of k and n. However, this is just the tip of the iceburg of applications of modular forms; other applications include: congruences of partition numbers, counting sums of two and four squares, counting solutions to quadratic forms with error terms, proving the irrationality of $\zeta(3)$, computing class numbers, cryptography, and much much more. Students who choose this project should pick and explain some interesting applications from the reference below.

Source: Zagier, *Elliptic modular forms and their applications*.

3 Automorphic forms

Automorphic forms generalize both modular forms (which are the $G = \mathrm{GL}_2$ case of automoprhic forms) and the Hecke characters of class field theory (the GL_1 case). Students who chose this topic should give an introduction to automorphic forms and explain how modular forms arise as a special case. It may be useful to have some prior familiarity with the adele ring of $\mathbb{Q}$.

Suggested source: Gelbart, *An elementary introduction to the Langlands program*.

4 Congruences of modular forms and p-adic modular forms

Fourier expansions of modular forms often have interesting properties modulo a prime p. For example, if E_k denotes the Eisenstein series normalized so that its Fourier expansion has constant term 1, then when $p-1|k$ all remaining terms are congruent to 0 mod p. This lead Serre to develop a fruitful theory of congruences between modular forms and p-adic modular forms. Students who pick this topic should explain the basic congruence properties of Eisenstein series and give an introduction to Serre's theory of p-adic modular forms.

Suggested source: Serre's paper is excellent for students willing to read some (very readable) mathematical French. Other references include chapter 2 of this.

5   Jacobian varities and the Galois representation associated to a modular form

Towards the end of the course, we will talk about, but not construct, the Galois representation asssociated to a modular form. The construction of said Galois representation involves forming an abelian variety associated to the Fourier coefficients of a modular forms, and then taking its Tate module. Students who choose this topic should explain the construction of this abelian variety and explain how the Galois representation on its Tate module relates to the Fourier expansion of the modular form.
Suggested source: Diamond and Shurman, §6.3 and §6.6.

6   Katz modular forms

We defined modular forms as differentials on a space which parametrizes pairs (E, σ), where E is and elliptic curve, and Σ is some sort of extra structure (such as a choice of torsion point). There is another, perhaps more satisfying approach: one can define modular forms as functions on the set of triples (E, ω, Σ) where E is an elliptic curve, ω is a holomorphic differential on E, and Σ is the same sort of extra structure as before. The benefit of doing this is that it leads to a more geometric description of modular forms. In particular, one can replace $\mathbb{C}$ with other fields K (even fields like $\mathbb{F}_p$ and $\mathbb{Q}_p$) by considering K-valued functions on triples (E, ω, Σ), where E must now be defined over K. Students who choose this topic should explain Katz's definition of modular forms and its relationship to the usual definition. **Warning: this topic is more advanced than the others.**
Suggested source: Calegari, *Congruences between modular forms*; chapter 1, especially §1.2. Additional references are likely to be useful.

7   Modular curves as algebraic curves

We have seen that elliptic curves can be viewed both as Riemann surfaces and algebraic curves. The situation is similar for modular curves, which we defined only as Riemann surfaces. Moreover, as algebraic curves, the modular curves are defined over $\mathbb{Q}$ or a cyclotomic extension of $\mathbb{Q}$. Students who chose this topic should present a proof of this fact.
Suggested source: Diamond and Shurman, §7.5 – 7.7.

8   Representable functors and modular curves as moduli spaces

We have seen that there are bijections between points on a modular curve and isomorphism classes of elliptic curves with extra structure. A geometric object which parametrizes something in this way is called a *coarse moduli spaces*, and modular curves give an example. However, there is a stronger, more categorical, sense in which a geometric object can parametrize something; such geometric objects are called *fine moduli spaces*, and only some modular curves fall into this category. Students who chose this topic should explain the basic notion of a fine moduli space and how elliptic curves with nontrivial automorphisms cause some modular curves to fail to be fine moduli spaces.
Suggested source: §3.2 of these notes gives a nice explanation of representable functors. For the

modular curve as a moduli space, section §3.3 of the same notes are good for students who have seen schemes; more elementary references include §2 of this.