$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$$

$x^s = e^{s\log x}$ $\qquad |n^s| = n^\sigma$

$s>1$

$s=\sigma+it$

$s:\ \sigma>1.$

$$\prod_p \left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\cdots\right) = \sum_{n\geq 1}\frac{1}{n^s} = \zeta(s)$$

$$\frac{1}{p_1^{se_1}p_2^{se_2}\cdots p_k^{se_k}} = \frac{1}{n^s}$$

$n = p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$

$\frac{1}{1-p^{-s}}$

$$\therefore \boxed{\zeta(s) = \prod_p \frac{1}{1-p^{-s}}}$$

$s=\sigma+it$, $\sigma>1$

$a_n = p_n^{-s}$

Marcus Num

Lemma 1. Let $\{a_n\}_{n\geq 1} \subseteq \mathbb{C}$ st

i) $\forall n: |a_n|<1$ ii) $\sum_{n\geq 1} |a_n| < \infty$.

Then $\prod_{n=1}^{\infty}(1-a_n)$ converges absolutely to a finite nonzero limit and

$$\prod_{n=1}^{\infty}(1-a_n)^{-1} = \sum_{j=0}^{\infty}\sum_{(r_1,\dots,r_j)} a_1^{r_1}\cdots a_j^{r_j} \quad (*)$$

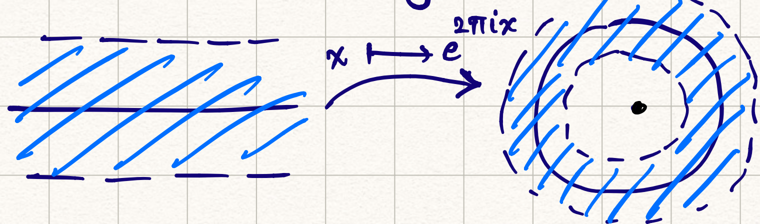
Sum is over all j-tuples $(r_1,\dots,r_j)$ of nonnegative integers with $r_j\geq 1$. The sum & product of $(*)$ are both absolutely convergent & hence independent of the order of the a_n.

• $\zeta(s) = \prod_p (1-p^{-s})^{-1}$ by UF in $\mathbb{Z}$

• show no zero s with $\sigma>1$

"Recall" Basic Theory of Fourier & Mellin:

Suppose $f:\mathbb{R}\longrightarrow\mathbb{C}$ differentiable & periodic with period 1. If f is sufficiently "nice" $f(x) = \tilde{f}(e^{2\pi i x})$



$$\begin{tikzcd} \mathbb{R} \arrow[r, "f"] \arrow[dr, "(x\mapsto e^{2\pi i x})"'] & \mathbb{C} \\ & \mathbb{C} \arrow[u, "\tilde{f}"'] \end{tikzcd}$$

$\tilde{f}(q) = \sum^{\infty} c_n q^n$ $\quad \therefore \boxed{f(x) = \sum^{\infty} c_n e^{2\pi i x n}}$ $\quad$ Fourier Series

c_n - Fourier coefficients. Obtained by $\boxed{c_n = \int_0^1 f(x) e^{-2\pi i x n} dx}$

Important Lemma 2: If $f \in \mathcal{C}^2(\mathbb{R})$ Then fourier series $\xrightarrow{abs} f$.

i) Locally Lipschitz $\to \alpha$-Hölder for some $\alpha \in (0,1]$

ii) Periodic ∴ Same as values on compact $\mathbb{R}/\mathbb{Z}$ $\to$ bounded.

∴ Wiegner.

Notation:

i) / :
ii) $\mathcal{C}$:

$\mathbb{R} \to \mathbb{C}$ periodic period ω

$\mathbb{R}/\mathbb{Z}\omega \to \mathbb{C}$

$\mathbb{R}/\mathbb{Z} \to \mathbb{C}$

$\mathcal{C}^0 \quad \mathcal{C}^1 \quad \mathcal{C}^2 \quad \mathcal{C}^\infty$

$f: \mathbb{R} \to \mathbb{C}$ not necessarily periodic but $\mathcal{L}^1(\mathbb{R})$ i.e. $\int_{\mathbb{R}} |f| d\mu < \infty$.

Define the Fourier Transform

Complex variable ξ

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx$$

· Time vs Fr.

Thm 1. (Poisson Summation Formula) Let $f: \mathbb{R} \to \mathbb{C}$ be $\mathcal{C}^2$ s.t. $\exists r > 1 : (|x|^r + 1)(|f(x)| + |f''(x)|) = O(1)$,

$f = O(g)$

$|f| \le B|g|$

Then

$$\sum_{m=-\infty}^{\infty} f(m) = \sum_{n=-\infty}^{\infty} \hat{f}(n)$$

where both sums absolutely.

Pf. Define $F: \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ by $F(x) = \sum_{m=-\infty}^{\infty} f(x+m)$; $\xrightarrow{abs}$ to $\mathcal{C}^2$ function.

∴ By Lemma 2, Fourier Series $\xrightarrow{abs} F$. In particular

$$\sum_m f(m) = F(0) = \sum_{n=-\infty}^{\infty} \left(\int_0^1 F(x) e^{-2\pi i x n} \right) \cancel{e^{2\pi i 0}}$$

$F(0) = \sum c_n \xrightarrow{abs}$

But now, $\int_0^1 F(x)e^{-2\pi i x n}\,dx = \int_0^1 \sum_{m=-\infty}^{\infty} f(x+m)e^{-2\pi i x n}\,dx$

$$= \sum_{m=-\infty}^{\infty} \int_0^1 \underline{f(x+m)}\, e^{-2\pi i x n}\,dx = \sum_{m=-\infty}^{\infty} \int_m^{m+1} f(x)e^{-2\pi i x n}\,dx = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x n}\,dx$$

$$\sum_m f(m) = F(0) = \sum_n \hat{f}(n). \qquad = \underline{\hat{f}(n)}. \quad \blacksquare$$

Another transform: Mellin.

$f: \mathbb{R}_{>0} \to \mathbb{C}$ st "nice"

i) $x \to \infty$: decays fast enough

ii) $x \to 0^+$: doesn't blow up $+\infty$ fast

Define

$$(\mathcal{M}f)(s) = \int_0^\infty x^{s-1} f(x)\,dx = \int_0^\infty f(x)\,x^s\,\frac{dx}{x} \qquad (\text{For } \sigma>0.)$$

IMPORTANT EXAMPLE. Take $f(x) = e^{-x}$

Then $\mathcal{M}f(s) = \underbrace{\int_0^\infty x^{s-1}e^{-x}\,dx}_{\text{converges for } \sigma>0.} = \boxed{\Gamma(s)}$

Observe: i) $\Gamma(1) = \int_0^\infty e^{-x}\,dx = -e^{-x}\Big]_0^\infty = 1 \quad \therefore \boxed{\Gamma(1)=1}$

$\boxed{\sigma\Gamma(\sigma) = \Gamma(\sigma+1)}$ ii) If $\sigma>0$, $\sigma\Gamma(\sigma) = \int_0^\infty e^{-x}\,d(x^\sigma) =$ [redacted] $= \int_0^\infty x^\sigma d(e^{-x}) = \Gamma(\sigma+1)$

iii) Consider function $s\Gamma(s) - \Gamma(s+1)$ for $\mathrm{Re}(s)>0$.

$\underline{\Gamma(s+1) = s\Gamma(s)}$ for $\mathrm{Re}(s)>0$.

Define $\Pi(s) = \Gamma(s+1)$. Then $\underline{\Pi(0)=1}$ & $\Pi(s) = s\Pi(s-1)$.

$\therefore$ By induction, voilà! $\Pi(n) = n!$ $\quad n \geq 0$

$\boxed{\Gamma(n) = (n-1)!}$ $\quad n \geq 1$

$\Pi(0) = 1$

$\Pi(n) = n\,\Pi(n-1)$

$\longrightarrow$ Euler Mascheroni const.

• Bohr-Mollerup

• Euler-Masch.

Turns out, $\Gamma(s) = \dfrac{e^{-\gamma s}}{s} \prod_{n=1}^{\infty} \left(1+\dfrac{s}{n}\right)^{-1} e^{s/n}$

$\gamma = \lim_{n\to\infty} (H_n - \log n)$

$1+\frac{1}{2}+\dots+\frac{1}{n}$

$\simeq 0.577\ldots$

· Merci!

· No zeroes, only ∞s.

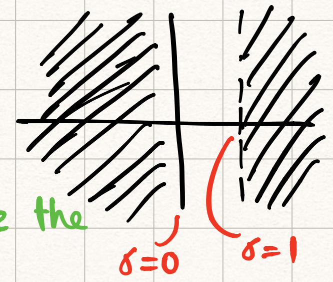
· Residues?

Meromorphic on all of $\mathbb{C}$

$0, -1, -2, -3, \ldots$

Main Theorem. If $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ then ξ extends to $\xi \in \mathrm{Mer}(\mathbb{C})$, holomorphic everywhere except for simple poles at $s=0,1$.
Further, "reflection formula" $\xi(s) = \xi(1-s)$.

Cor.1. ζ extends to $\mathrm{Mer}(\mathbb{C})$ with simple pole at $s=1$. It has zeroes at $s \in -2\mathbb{N}$, the "trivial" zeroes, and no other outside the CRITICAL STRIP $0 \le \sigma \le 1$. $\mathrm{Re}(s) \in [0,1]$



$\zeta = \xi(s) / \pi^{-s/2}\Gamma(s/2)$ $\quad \xi(s) \quad s=-2n \quad n \in \mathbb{N}_0$

$|R|=1$

Pf. Jacobi Theta Function $\Theta(q) = \sum_{-\infty}^{\infty} q^{n^2} = 1 + 2\sum_{n=1}^{\infty} q^{n^2}$ $\quad (\Theta(q))^k = \sum s_k(n) q^n$

By Hadamard, $\Sigma \xrightarrow{abs} \Theta$ for $|q|<1$

Consider the function $\theta(u) = \Theta(e^{-\pi u}) = \boxed{1 + 2\sum_{n=1}^{\infty} e^{-\pi u n^2}}$

By above, converges for $\sigma > 0$.

Lemma. $\theta(1/u) = u^{1/2}\theta(u)$ for $u \in \mathbb{R}_{>0}$.

Pf.

Consider function $f: \mathbb{R} \to \mathbb{C}$ by $f(x) = e^{-\pi u x^2}$ $(u>0)$.

Satisfies conditions of Theorem1 $\therefore$ $\theta(u) = \sum_m f(m) = \sum_n \hat{f}(n)$.

Now $\hat{f}(\xi) = \int^{\infty} f(x) e^{-2\pi i x \xi} dx = \int^{\infty} e^{-\pi u (x^2 + 2x \cdot i\xi/u)} dx$

$$\therefore\ \hat f(\xi) = e^{-\pi u^{-1}\xi^2}\int_{-\infty}^{\infty} e^{-\pi u(x+i\xi/u)^2}\,dx = e^{-\pi u^{-1}\xi^2}\cdot\int_{-\infty+hi}^{\infty+hi} e^{-\pi u z^2}\,dz$$

CONTOUR INTEGRATION! Let $z = x + i\xi/u$ and $h = \mathrm{Im}(i\xi/u)$.

Consider the contour $\partial\Omega$ where

But holomorphic every wh

By Residue Theorem, $\oint_{\partial\Omega} e^{-\pi u z^2}\,dz = 2\pi i \sum_{a\in\Omega} \operatorname{res}_{z=a}\left(e^{-\pi u z^2}\right) = \boxed{0}$

$$\therefore \int_{-R}^{R} e^{-\pi u x^2}\,dx - \int_{-R+hi}^{R+hi} e^{-\pi u z^2}\,dz + \cancel{\int_{R}^{R+hi} e^{-\pi u z^2}\,dz} + \cancel{\int_{-R}^{-R+hi} e^{-\pi u z^2}\,dz} =$$

But now $\left|\int_{R}^{R+hi} e^{-\pi u z^2}\,dz\right| \le \underbrace{|hi|}_{h}\cdot \underbrace{\sup_{\substack{z=R+yi\\ y \text{ b/w } 0\,\&\,h}} \left|e^{-\pi u z^2}\right|}_{\sim\, e^{-\pi u R^2}}$ $\therefore \lim_{R\to\infty}$ gives 0.

$$\therefore\ \hat f(\xi) = e^{-\pi u^{-1}\xi^2}\cdot\int_{-\infty}^{\infty} e^{-\pi u x^2}\,dx$$

Gaussian Integral $= 1/\sqrt{u} = u^{-1/2}$

$\therefore$ By Poisson Summation, $\theta(u) = \sum_m f(m) = \sum_n \hat f(n) =$

$$\theta(u) = \sum_m f(m) = u^{-1/2}\underbrace{\sum_n e^{-\pi u^{-1} n^2}}_{\theta(u^{-1})}$$

■

$\theta(u) = u^{1/2}\theta(u)$

Main Proof. $\theta : \mathbb{R}_{>0} \longrightarrow \mathbb{C}$.

The $M\left(\frac{\theta(u)-1}{2}\right) = \int_0^\infty \left(\sum_{n=1}^{\infty} e^{-\pi u n^2}\right) u^{s-1}\, du = \sum_{n=1}^{\infty} \int_0^\infty u^{s-1} e^{-\pi u n^2} du$

(For $\sigma > 0$.)

By changing $v = \pi u n^2$ $\quad u = v/\pi n^2$

$$\frac{1}{\pi^{s-1} n^{2(s-1)}} \int_0^\infty v^{s-1} \cdot e^{-v} \cdot \frac{dv}{\pi n^2} = \frac{1}{(\pi n^2)^s} \underbrace{\int_0^\infty v^{s-1} e^{-v} dv}_{} = \frac{\Gamma(s)}{\pi^s n^{2s}}$$

$$\pi^{-s}\Gamma(s) \sum_{n=1}^{\infty} \frac{1}{n^{2s}} = \pi^{-s}\Gamma(s)\zeta(2s) = \xi(2s). \quad M\left(\frac{\theta(u)-1}{2}\right) = \xi(2s)$$

$$\therefore\ 2\xi(s) = \int_0^\infty (\theta(u)-1) u^{s/2} \frac{du}{u} = \underbrace{\int_0^1 (\theta(u)-1) u^{s/2} \frac{du}{u}}_{} + \int_1^\infty (\theta(u)-1)\, u^{s/2} \frac{du}{u}$$

Now $\displaystyle\int_0^1 (\theta(u) - 1)\, u^{s/2} \frac{du}{u} = -\frac{2}{s} u^{s/2}\Big|_0^1 + \int_0^1 \theta(u)\, u^{s/2} \frac{du}{u}$

$$= -\frac{2}{s} + \underbrace{\int_0^1 \theta(u)\, u^{s/2} \frac{du}{u}}_{}$$

Substitute $v = 1/u$. $\quad du = \frac{-1}{v^2} dv$

$$\int_1^\infty \underline{\underline{\theta\left(\tfrac{1}{v}\right)}}\, v^{-s/2}\, dv \cdot \cancel{v} \cdot \frac{1}{v^{\cancel{2}}}$$

$$\theta(1/v) = v^{1/2}\theta(v) \qquad \int_1^\infty \theta(v)\, v^{\left(\frac{1-s}{2}\right)} \frac{dv}{v}$$

$$2\xi(s) = -\frac{2}{s} + \underbrace{\int_1^\infty \underline{\theta(v)}\, v^{\frac{1-s}{2}} \frac{dv}{v}}_{} + \int_1^\infty (\theta(u)-1)\, u^{s/2} \frac{du}{u}$$

If also $\sigma < 1$.

∞

Therefore, $\boxed{\xi(s) + \frac{1}{s} + \frac{1}{1-s} = \frac{1}{2}\int_1^\infty (\theta(u)-1)\left(u^{s/2} + u^{(1-s)/2}\right)\frac{du}{u}}$ $u \in \mathbb{R}_{>0}$

Observe that RHS $\in \mathrm{Hol}(\mathbb{C})$, because on any compact $K \subseteq \mathbb{C}$

$$R = \max_{s \in K}\left\{|s/2|, \left|\tfrac{1-s}{2}\right|\right\} < \infty$$

$$\therefore \left|\int_1^\infty \text{"} \, du\right| \le \int_1^\infty |\theta(u)-1|\left|u^{s/2} + u^{\frac{1-s}{2}}\right|\frac{du}{u}$$

$$\left|u^{s/2} + u^{\frac{1-s}{2}}\right| \le |u^{s/2}| + |u^{\frac{1-s}{2}}| \le 2u^R$$

$$\le 2\int_1^\infty |\theta(u)-1| \, u^{R-1} du < \infty$$

decays exponential

$\therefore$ Converges $\xrightarrow{\text{abs}}$ on $K \subseteq \mathbb{C}$. ∎

$$\boxed{\xi(s) + \frac{1}{s} + \frac{1}{1-s}} \in \mathrm{Hol}(\mathbb{C})$$ ✓

CREDITS: i) TO SAM, OFC.

ii) MATH 259 NOTES BY PROF. ELKIES.

now called 229X.

iii) Ahlfoss.