Last time

Ramification $\rho : G_{\mathbb{Q}} \longrightarrow GL_n(F)$

$$I_p \leq D_p \leq G_{\mathbb{Q}} \qquad p \text{ prime}$$

$$\begin{tikzcd} 1 \arrow[r] & I_p \arrow[r] & D_p \arrow[r] & G_{\mathbb{F}_p} \arrow[r] & 1 \\ & & \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \arrow[u, "\wr\wr ?"] \arrow[ur] & & \end{tikzcd}$$

with $\sigma_p \in G_{\mathbb{F}_p}$.

ρ is <u>unramified</u> at p if $I_p \subseteq \ker\rho$

if so, $\rho(\sigma_p) \in GL_n(F)$ makes sense

There are many choices for D_p, all conjugate in $G_{\mathbb{Q}}$

$\rho(\sigma_p)$ is well-defined up to conjugacy

L-functions

2 ways of forming them from representation theoretic data:

- Artin L-functions: come from $\rho : G_{\mathbb{Q}} \longrightarrow GL_n(\mathbb{C})$ (<u>complex</u>)
- L-functions of compatible systems of ℓ-adic representations: $\{\chi_\ell\}_{\ell \text{ prime}}$

<u>Examples</u>: $\zeta(s) = \sum_{n \geq 1} n^{-s} = \prod_p \frac{1}{1-p^{-s}}$

$$L(\chi, s) = \sum_{n\geq 1} \chi(n) n^{-s} = \prod_{p \nmid N} \frac{1}{1-\chi(p)p^{-s}} \qquad \chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$$

Defined on $\{s \in \mathbb{C} : \mathrm{Re}(s) > 1\}$

$\Big\{$ Analytic continuation to $\mathbb{C}$, functional equation relating $\zeta(s) \longleftrightarrow \zeta(1-s)$

deeper properties.

Artin L-functions

Let $\rho : G_{\mathbb{Q}} \longrightarrow GL_n(\mathbb{C})$.

Recall: ρ factors as $G_{\mathbb{Q}} \twoheadrightarrow \mathrm{Gal}(K/\mathbb{Q}) \rightarrow GL_n(\mathbb{C})$

$K/\mathbb{Q}$ fin. Gal.

So instead let $\rho : \mathrm{Gal}(L/K) \longrightarrow GL(V)$

L/K is fin. Gal, $V/\mathbb{C}$ n-dim'l v. space.

$$\begin{array}{cc} L & \mathfrak{P} \\ | & | \\ K & \mathfrak{p} \end{array}$$

Defn

(1) If $I_{\mathfrak{P}} \le \mathrm{Gal}(L/K)$ is a choice of inertia group, let

$$V^{I_{\mathfrak{P}}} = \{ v \in V : \rho(\sigma) v = v \quad \forall \sigma \in I_{\mathfrak{P}} \}.$$

ρ gives a rep'n $\rho^{I_{\mathfrak{P}}}$ of $\mathrm{Gal}(L/K)$ on $V^{I_{\mathfrak{P}}}$ unramified at $\mathfrak{p}$.

(2) For each prime $\wp$, the <u>local L-factor</u> is

$$L_\wp(\rho, T) = \frac{1}{\det(1 - T\sigma_\wp \mid \rho^{I_\wp})} \in \mathbb{C}(T)$$

If $\chi_{\rho^{I_\wp}(\sigma_\wp)}(T) \in \mathbb{C}[T]$ is the char poly of $\sigma_\wp$ acting on $V^{I_\wp}$, then

$$\det(1 - T\sigma_\wp \mid \rho^{I_\wp}) = \chi_{\rho^{I_\wp}(\sigma_\wp)}\left(\tfrac{1}{T}\right) \cdot T^{\deg \chi}$$

so this is independent of the choice of $\sigma_\wp$.

(3) The <u>(global) L-function</u> is

$$L(\rho, s) = \prod_\wp L_\wp(\rho, N\wp^{-s})$$

$$\left(N\wp = \text{norm of } \wp = \#(\mathcal{O}_K/\wp) \in \mathbb{Z} \right)$$

for $s \in \mathbb{C}$ s.t. this product converges.

<u>Rmk</u> If $\rho : G_\mathbb{Q} \to GL_n(\mathbb{C})$, we write $L(\rho, s)$ for $L(\rho', s)$, any ρ' s.t.

$$\begin{tikzcd} G_\mathbb{Q} \arrow[r, "\rho"] \arrow[d] & GL_n(\mathbb{C}) \\ \mathrm{Gal}(K/\mathbb{Q}) \arrow[ur, "\rho'"'] & \end{tikzcd}$$

(check that this doesn't depend on choice of p').

Ex. $\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_1(\mathbb{C})$.

ρ ramifies @ no prime

$$L_p(\rho, T) = \frac{1}{\det(1 - T\sigma_p \mid \rho)} = \frac{1}{1-T}$$

$$L(\rho, s) = \prod_p \frac{1}{1-p^{-s}} = \zeta(s).$$

Ex. $\rho : \mathrm{Gal}(\bar{K}/K) \longrightarrow GL_1(\mathbb{C})$.

ρ unramified everywhere

$$L_{\mathfrak{p}}(\rho, T) = \frac{1}{1-T}$$

$$L(\rho, s) = \prod_{\mathfrak{p}} \frac{1}{1-N\mathfrak{p}^{-s}} = \sum_{I \subseteq \mathcal{O}_K} NI^{-s}$$

$$= \zeta_K(s)$$

Ex. Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ be a Dirichlet character.

$$\begin{tikzcd} G_{\mathbb{Q}} \arrow[r, two heads] \arrow[rrr, bend right, "\rho_\chi"'] & \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \arrow[r, "\sim"] & (\mathbb{Z}/N\mathbb{Z})^\times \arrow[r, "\chi"] & \mathbb{C}^\times \end{tikzcd}$$

ρ_χ is unramified at $p \nmid N$

$$L_p(\rho, s) = \frac{1}{\det(1 - T\mathrm{Fr}_p | \rho)} = \frac{1}{1 - T\chi(p)}$$

Problem: it's possible that $p \nmid N$ but ρ_χ is unram'd at p.

Recall: a Dirichlet char $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}$ is imprimitive if it factors as

$$\begin{tikzcd} \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \arrow[r, "\cong"] \arrow[d] & (\mathbb{Z}/N\mathbb{Z})^\times \arrow[rr, "\chi"] \arrow[dr] & & \mathbb{C}^\times \\ \mathrm{Gal}(\mathbb{Q}(\zeta_M)/\mathbb{Q}) \arrow[rr, "\cong"] & & (\mathbb{Z}/M\mathbb{Z})^\times \arrow[ur, "\chi_0"'] & \end{tikzcd}$$

for $M \mid N$

Note $\rho_\chi = \rho_{\chi_0}$ but possibly $L(\chi, s) \neq L(\chi_0, s)$

Suppose that χ is primitive (so $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ is the smallest such that ρ_χ factors through.

By CFT, ρ_χ is ramified at all $p \nmid N$.

So $V^{I_p} = 0$.

$$L_p(\rho_\chi, T) = \frac{1}{\det(1 - T\sigma_p \mid \rho^{I_p})} = \frac{1}{1} = 1$$

$$L(\rho_\chi, s) = \prod_{p \nmid N} \frac{1}{1 - \chi(p)p^{-s}} = L(\chi, s).$$

Rmk

We've shown $\left\{ \begin{array}{c} \text{primitive} \\ \text{Dirichlet chars} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{1-dim'l complex} \\ \text{Gal reps} \end{array} \right\}$

preserving L-functions.

CFT gives

$$\{\text{Hecke chars}\} \xleftrightarrow{\substack{\text{preserving} \\ \text{L-funcs}}} \{\text{Dirichlet chars}\}$$

Thm Let $\rho : \mathrm{Gal}(L/k) \to GL_n(\mathbb{C})$ be a Gal. rep.

There exists a function $L_\infty(\rho, s)$ and an integer $c(\rho)$ such that the completed L-function

$$\Lambda(\rho, s) = c(\rho)^{s/2} L_\infty(\rho, s) L(\rho, s)$$

has a meromorphic continuation to $\mathbb{C}$, and

$$\Lambda(\rho, s) = W(\rho) \Lambda(\bar{\rho}, 1-s)$$

where $W(\rho) \in S^1 \subseteq \mathbb{C}^\times$.

L-functions of compatibles systems of l-adic reps

χ_l : l-adic cyclo char.

Good facts:

- χ_l is unram'd @ $p \neq l$.
- $\chi_l(\sigma_p) = p \in \mathbb{Q}_l^\times$ is in $\mathbb{Q}$ and is independent of $l \neq p$.

  $\det(1 - T\sigma_p \mid \chi_l) \in \mathbb{Q}_l[T]$ is in $\mathbb{Q}[T]$ and independent of l.

Defn Let $\rho_\ell : G_{\mathbb{Q}} \longrightarrow GL_n(\mathbb{Q}_\ell)$ be a Gal rep for each prime ℓ. Let $\rho = \{\rho_\ell\}_\ell$.

ρ is a compatible system if

(1) $\exists$ finite set S of primes s.t. ρ_ℓ is unram'd at all $p \notin S \cup \{\ell\}$ for all ℓ.

(2) $\det(1 - T\sigma_p \mid \rho_\ell^{I_p}) \in \mathbb{Q}_\ell[T]$ is in $\mathbb{Q}[T]$, and is independent of $\ell \neq p$.

Defn The L-function of $\rho = \{\rho_\ell\}$:

$$L_p(\rho, T) = \frac{1}{\det(1 - T\sigma_p \mid \rho_\ell^{I_p})} \quad \text{for any } \ell \neq p.$$

$$L(\rho, s) = \prod_p L_p(\rho, p^{-s}).$$

Ex. $\chi = \{\chi_\ell\}_\ell$.

$$L_p(\chi, T) = \frac{1}{\det(1 - T\sigma_p \mid \chi_\ell)} = \frac{1}{1 - pT}.$$

$$L(\chi, s) = \prod_p \frac{1}{1 - p \cdot p^{-s}} = \prod_p \frac{1}{1-p^{1-s}} = \zeta(s-1).$$

__Rmk__ Compatible systems come from geometry.

__Rmk__ What we want is a <u>global rep</u>

$$\begin{tikzcd} \rho : G_{\mathbb{Q}} \arrow[r] & GL_n(\mathbb{Q}) \arrow[d, hook] \\ & GL_n(\mathbb{Q}_\ell) \end{tikzcd}$$

This gives a compatible system, but not conversely.