Ramification

$$\rho : G_{\mathbb{Q}} \longrightarrow GL_n(K)$$

$$\sigma_p = (x \longmapsto x^p) \in G_{\mathbb{F}_p}$$

$\rho(\sigma_p) \in GL_n(K)$ well-defined up to conjugation.

Want decomposition group $D_p \leq G_{\mathbb{Q}}$ s.t. $D_p \twoheadrightarrow G_{\mathbb{F}_p}$.

$$1 \longrightarrow I_p \longrightarrow D_p \longrightarrow G_{\mathbb{F}_p} \longrightarrow 1$$

I_p: inertia group; $\sigma_p \in G_{\mathbb{F}_p}$

if $I_p \subseteq \ker \rho$, then $\rho(\sigma_p)$ makes sense.
If so we say that ρ is <u>unramified</u> at p.

Some choices are made in defining D_p
Different choice lead to conjugate D_p's.

Two ways of defining D_p:

<u>Way 1</u> (w/o details)

$$\begin{array}{cc} \overline{\mathbb{Q}} & \mathfrak{p} \\ | & | \\ \mathbb{Q} & p \end{array}$$

$$D_p = \{\sigma \in G_{\mathbb{Q}} : \sigma(\mathfrak{p}) = \mathfrak{p}\}$$

$$\begin{array}{ccc} \overline{\mathbb{Z}} & \xrightarrow{\sigma} & \overline{\mathbb{Z}} \\ \downarrow & & \downarrow \\ \overline{\mathbb{F}}_p & \xrightarrow{\overline{\sigma}} & \overline{\mathbb{F}}_p = \overline{\mathbb{Z}}/\mathfrak{p} \end{array}$$

This gives a map $D_p \twoheadrightarrow G_{\mathbb{F}_p}$

Let I_p be the kernel.

Gives

$$1 \longrightarrow I_p \longrightarrow D_p \longrightarrow G_{\mathbb{F}_p} \longrightarrow 1.$$

Different choices of $\mathfrak{p} | p$ gives conjugate D_p's, therefore conjugate I_p's

Since $\ker \rho$ is normal, $I_p \subseteq \ker\rho$ for one choice of $\mathfrak{p}|p$ iff $I_p \subseteq \ker\rho$ for all choices.

<u>Way 2</u> (with some details)

$$\begin{tikzcd}
\overline{\mathbb{Q}} \arrow[r, hook, "\exists"] & \overline{\mathbb{Q}_p} \\
\mathbb{Q} \arrow[u, hook] \arrow[r, hook] \arrow[ur] & \mathbb{Q}_p \arrow[u, hook]
\end{tikzcd}$$

$$\begin{array}{l} M \\ | \\ L \\ | \\ K \end{array}$$

$\mathrm{Gal}(M/K) \twoheadrightarrow \mathrm{Gal}(L/M)$

$$\begin{tikzcd}
M \arrow[r, hook, "\exists"] & \overline{K} \\
L \arrow[u, hook] \arrow[ur, hook]
\end{tikzcd}$$

This gives (non-uniquely) an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$

The set of such embeddings is acted on by $G_{\mathbb{Q}}$

$$\sigma \in G_{\mathbb{Q}}: \quad \begin{tikzcd} \overline{\mathbb{Q}} \arrow[d, hook] \\ \overline{\mathbb{Q}_p} \end{tikzcd} \longmapsto \begin{tikzcd} \overline{\mathbb{Q}} \arrow[r, "\sigma"] \arrow[dr, bend right] & \overline{\mathbb{Q}} \arrow[d, hook] \\ & \overline{\mathbb{Q}}_p \end{tikzcd}$$

This action is transitive.

(Picture: an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$ "is" a prime $\mathfrak{p}$ of $\overline{\mathbb{Q}}$ over p).

Fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$.

$$\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \hookrightarrow \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}) \xrightarrow[\text{injective!}]{} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$$

$G_{\mathbb{Q}_p}$
$\cong$
Let $D_p \leq G_{\mathbb{Q}}$ be the image of this injection.

We have a map

$$\begin{array}{ccc} G_{\mathbb{Q}_p} & \twoheadrightarrow & G_{\mathbb{F}_p} \\ \| & & \| \\ \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) & \to & \mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \end{array}$$

Gives a map $D_p \twoheadrightarrow G_{\mathbb{F}_p}$, giving

$$1 \to I_p \to D_p \to G_{\mathbb{F}_p} \to 1.$$

The embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$ are conjugate (acted on transitively by $G_{\mathbb{Q}}$), and conjugate embeddings give conjugate D_p's.

<u>Rmk</u>

$$\begin{array}{ccc} & K & \mathfrak{p} \\ \text{fin.} & | & | \\ & \mathbb{Q} & p \end{array}$$

(choice of $\mathfrak{p}$ is like a choice of $K \hookrightarrow \overline{\mathbb{Q}_p}$)

$$\begin{tikzcd} K \arrow[rr, hook] \arrow[dr] & & \overline{\mathbb{Q}_p} \\ & K_{\mathfrak{p}} \arrow[ur, hook] & \end{tikzcd}$$

$$\begin{array}{ccc} \mathrm{Gal}(K_{\mathfrak{p}}/\mathbb{Q}_p) & \longrightarrow & \mathrm{Gal}(K/\mathbb{Q}) \\ \cup\!\!\!\!\mid & & \cup\!\!\!\!\mid \\ \sigma & \longmapsto & \sigma|_K \end{array}$$

σ, σ^{-1} are continuous on $K_{\mathfrak{p}}$

$\Longrightarrow \sigma(\mathfrak{p}) = \mathfrak{p}$

so

$$\begin{array}{ccc} \mathrm{Gal}(K_{\mathfrak{p}}/\mathbb{Q}_p) & \xrightarrow{\sim} & D_{\mathfrak{p}} = \{\sigma \in \mathrm{Gal}(K/\mathbb{Q}) : \sigma(\mathfrak{p}) = \mathfrak{p}\} \\ \cup\!\!\!\!\mid & & \cup\!\!\!\!\mid \\ \hat{\sigma} & \xleftarrow[\text{continuously}]{\text{extend}} & \sigma \end{array}$$

Rmk

$$\begin{array}{c} K \\ \Big| \text{fin} \\ \mathbb{Q} \end{array}$$

$\{\text{primes } \mathfrak{p} | p\} \xleftrightarrow{\text{Ostrowski's Theorem}} \left\{ \begin{array}{c} \text{absolute values on } K \\ \text{extending } |\cdot|_p \text{ on } \mathbb{Q} \end{array} \right\}$

$\longleftrightarrow \{\text{embeddings } K \hookrightarrow \overline{\mathbb{Q}_p}\}$

Ostroski's Thm: nonarchimedean absolute values on K are $|\cdot|_{\mathfrak{p}}$ for primes $\mathfrak{p}$ of K.

$$\left\{ \begin{array}{c} \text{absolute values on } K \\ \text{extending } |\cdot|_p \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{embeddings} \\ K \hookrightarrow \overline{\mathbb{Q}_p} \end{array} \right\}$$

$$|\cdot|_{\mathfrak{p}} \longmapsto (K \hookrightarrow K_{\mathfrak{p}} \hookrightarrow \overline{\mathbb{Q}_p})$$

$$|\cdot|_{\overline{\mathbb{Q}_p}}|_K \longleftarrow\!\!\shortmid K \hookrightarrow \overline{\mathbb{Q}_p}$$

$$\begin{tikzcd} {|\cdot|} \arrow[r, mapsto] & (K \arrow[r, hook] & K_{|\cdot|} \arrow[r, hook, "*"] \arrow[d] & \overline{\mathbb{Q}_p}) \\ & & {|\cdot|} & \end{tikzcd}$$

Monday's lecture: analytic facts about $\zeta(s)$

- Proving meromorphic continuation
- Proving funct'l eqn.

<u>Prop</u> Let D_p be a choice of absolute decomposition group, $I_p \leq D_p$ be the inertia grp. Let $K/\mathbb{Q}$ fin. Gal. Then $D_p|_K$ is a decomp. group for $K/\mathbb{Q}$. Same for I_p.

<u>Pf.</u>

$$\begin{tikzcd} G_{\mathbb{Q}_p} \arrow[r, phantom, "\cong"] \arrow[d] & D_p \arrow[r, hook] \arrow[d] & G_{\mathbb{Q}} \arrow[d] \\ \mathrm{Gal}(K_{\mathfrak{p}}/\mathbb{Q}_p) \arrow[r, phantom, "\cong"] & D_{\mathfrak{p}} \arrow[r, hook] & \mathrm{Gal}(K/\mathbb{Q}) \end{tikzcd} \qquad \begin{tikzcd} K \arrow[d, dash] & \mathfrak{p} \arrow[d, dash] \\ \mathbb{Q} & p \end{tikzcd}$$

$$\begin{tikzcd} K \arrow[r, hook] \arrow[rr, bend right, "\mathfrak{p}"'] & \overline{\mathbb{Q}} \arrow[r, hook] & \overline{\mathbb{Q}_p} \end{tikzcd}$$

<u>Ex.</u> $\chi_\ell : G_{\mathbb{Q}} \longrightarrow \mathbb{Q}_\ell^\times$ ℓ-adic cyclo character

Unram'd at all $p \neq \ell$.

Let $p \neq \ell$ be prime.

$$I_p \big|_{\mathbb{Q}(\mu_{\ell^n})} \leq \mathrm{Gal}(\mathbb{Q}(\mu_{\ell^n})/\mathbb{Q}) \quad \text{is an}$$

inertia gp for $\mathbb{Q}(\mu_{\ell^n})/\mathbb{Q}$, hence trivial.

$$G_{\mathbb{Q}} \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}(\mu_{\ell^\infty})/\mathbb{Q}) \cong \varprojlim_{n\geq 1} \mathrm{Gal}(\mathbb{Q}(\mu_{\ell^n})/\mathbb{Q}) \to \varprojlim_{n\geq 1} (\mathbb{Z}/\ell^n\mathbb{Z})^\times \cong \mathbb{Z}_\ell^\times \hookrightarrow \mathbb{Q}_\ell^\times$$

$$\begin{array}{l} \cup\!\! \mid \\ I_p \longrightarrow\!\!\!\!\rightarrow \left(I_p|_{\mathbb{Q}(\mu_{\ell^n})} \right)_{n\geq 1} = \{1\}. \end{array}$$

$$\sigma_p \mapsto \left(\frac{p}{\mathbb{Q}(\mu_{\ell^\infty})/\mathbb{Q}} \right) \mapsto \left(\left(\frac{p}{\mathbb{Q}(\mu_{\ell^n})/\mathbb{Q}} \right) \right)_{n\geq 1} \mapsto (p)_{n\geq 1} \mapsto p \mapsto p$$

Frobenius at p
of $\mathbb{Q}(\mu_{\ell^\infty})/\mathbb{Q}$

$\chi_\ell(\sigma_p) = p \in \mathbb{Q}_\ell^\times \qquad \forall p \neq \ell.$

Note: χ_ℓ is unramified at almost every prime

$\chi_\ell(\sigma_p)$ does <u>not</u> depend on ℓ!

$\{\chi_\ell\}_\ell$ "compatible system"

<u>L-functions</u>

<u>Examples</u>

$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$

$\zeta(s) = \sum_{n\geq 1} n^{-s} \qquad L(\chi, s) = \sum_{n\geq 1} \chi(n)\, n^{-s}$

converge, holomorphic on $\mathrm{Re}(s) > 1$.

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}$$ characteristic polynomials of $\rho(\sigma_p)$

$$L(\chi, s) = \prod_{p \nmid N} \frac{1}{1 - \chi(p) p^{-s}}$$

Deeper facts: analytic continuation, functional eqn.

$L(\rho, s)$

Galois representation

2 dim'l Gal rep.

$$\frac{1}{1 - \_\, p^{-s} + \_\, p^{-2s}}$$