

$$\varprojlim_{M/\mathbb{F}_p} \text{Gal}(M/\mathbb{F}_p)$$

$$I = \{ M/\mathbb{F}_p \}$$

||

$$\varprojlim_{n \geq 1} \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$$

$$I' = \{ \mathbb{F}_{p^n}/\mathbb{F}_p \}$$

$$\hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z} \stackrel{\text{CRT}}{=} \prod_p \mathbb{Z}_p$$

$\hat{\phantom{x}}$  = "profinite completion"

$$\hat{G} = \varprojlim_{\substack{H \trianglelefteq G \\ G/H \text{ finite}}} G/H$$

Last time

Defn A Galois representation of dimension  $n$  over a (topological) field  $F$  is a continuous homomorphism

$$\rho: G_{\mathbb{Q}} \longrightarrow \text{GL}_n(F).$$

$$G_K = \text{Gal}(\overline{\mathbb{Q}}/K) \leq G_{\mathbb{Q}}$$

$K/\mathbb{Q}$  fin.

"Galois rep of  $G_K$ " = cont. hom.  $\rho: G_K \rightarrow \text{GL}_n(\mathbb{F})$ .

Ex. (1-adic cyclotomic character)

$$\mu_n = \{ \zeta \in \overline{\mathbb{Q}}^\times : \zeta^n = 1 \}$$

$$\begin{array}{c} \parallel \\ \langle \zeta_n \rangle \end{array}$$

$$\text{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$$

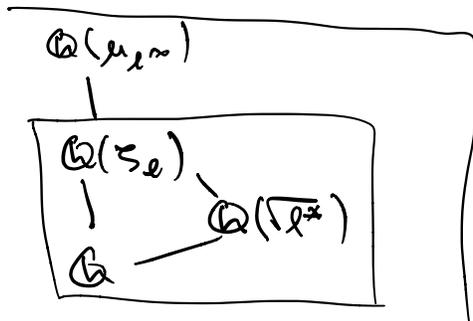
$$\begin{array}{ccc} \sigma & \longmapsto & a \\ \parallel & & \\ \zeta_n & \longmapsto & \zeta_n^a \end{array}$$

$$(\zeta_n \mapsto \zeta_n^a)$$

$$\text{Gal}(\mathbb{Q}(\mu_{\ell^\infty})/\mathbb{Q}) = \varprojlim_{n \geq 1} \text{Gal}(\mathbb{Q}(\mu_{\ell^n})/\mathbb{Q}) \cong \varprojlim_{n \geq 1} (\mathbb{Z}/\ell^n\mathbb{Z})^\times$$

$$\mathbb{Q}(\mu_{\ell^\infty}) = \bigcup_{n \geq 1} \mathbb{Q}(\mu_{\ell^n}) \cong \mathbb{Z}_\ell^\times$$

$$\chi_\ell: G_{\mathbb{Q}} \rightarrow \text{Gal}(\mathbb{Q}(\mu_{\ell^\infty})/\mathbb{Q}) \cong \mathbb{Z}_\ell^\times \hookrightarrow \mathbb{Q}_\ell^\times = \text{GL}_1(\mathbb{Q}_\ell)$$



$$\begin{array}{ccccccc}
 & & & G_{\mathbb{Q}} & \curvearrowright & & \\
 & & \subset & & \subset & & G_{\mathbb{Q}} \\
 \dots & \xrightarrow{x \cdot 1} & \mu_{\ell^3} & \xrightarrow{x \cdot 1} & \mu_{\ell^2} & \xrightarrow{x \cdot 1} & \mu_{\ell} \\
 & & \subset & & \subset & & \subset \\
 \dots & \longrightarrow & \mathbb{Z}/\ell^3\mathbb{Z} & \longrightarrow & \mathbb{Z}/\ell^2\mathbb{Z} & \longrightarrow & \mathbb{Z}/\ell\mathbb{Z}
 \end{array} \implies \begin{array}{c} \varprojlim_n \mu_{\ell^n} \\ \subset \\ \mathbb{Z}_{\ell} \end{array}$$

$T = \varprojlim_n \mu_{\ell^n}$  is a rank 1  $\mathbb{Z}_{\ell}$ -module with

a compatible  $G_{\mathbb{Q}}$  action

$T \otimes_{\mathbb{Z}_{\ell}} \mathbb{Q}_{\ell}$  rk 1  $\mathbb{Q}_{\ell}$  module w/  $G_{\mathbb{Q}}$ -action

$$\begin{array}{c} \Downarrow \\ \rho: G_{\mathbb{Q}} \longrightarrow GL(T \otimes_{\mathbb{Z}_{\ell}} \mathbb{Q}_{\ell}). \end{array}$$

"Forming a Tate module"

Suppose  $\rho: G_{\mathbb{Q}} \longrightarrow GL_n(\mathbb{C})$

$G_{\mathbb{Q}}$  and  $\mathbb{C}$  have very different topologies!

- $G_{\mathbb{Q}}$  has arbitrarily small subgps
- $GL_n(\mathbb{C})$  has "no small subgps"

Prop Let  $\rho: G_{\mathbb{Q}} \longrightarrow GL_n(\mathbb{C})$  be a complex

Gal rep. Then  $\rho$  factors as

$$G_{\mathbb{Q}} \longrightarrow \text{Gal}(K/\mathbb{Q}) \longrightarrow GL_n(\mathbb{C})$$

for some fin. Gal.  $K/\mathbb{Q}$ .

Pf. Let  $V \subseteq GL_n(\mathbb{C})$  open nbhd of  $1$  s.t.  
any  $H \subseteq GL_n(\mathbb{C})$  contained in  $V$  is trivial.

$\rho^{-1}(V) \subseteq G_{\mathbb{Q}}$  open nbhd of  $1$ .

So  $\exists K/\mathbb{Q}$  fin. Gal. extn s.t.

$$\text{Gal}(\bar{\mathbb{Q}}/K) \subseteq \rho^{-1}(V).$$

$$\rho(\text{Gal}(\bar{\mathbb{Q}}/K)) \subseteq V \Rightarrow \rho(\text{Gal}(\bar{\mathbb{Q}}/K)) = \{1\}.$$

Hence  $\text{Gal}(\bar{\mathbb{Q}}/K) \subseteq \text{Ker } \rho$ , so

$$\rho: G_{\mathbb{Q}} \rightarrow G_{\mathbb{Q}}/G_K = \text{Gal}(K/\mathbb{Q}) \rightarrow GL_n(\mathbb{C}). \square$$

### Ramification

$$\rho: G_{\mathbb{Q}} \rightarrow GL_n(K)$$

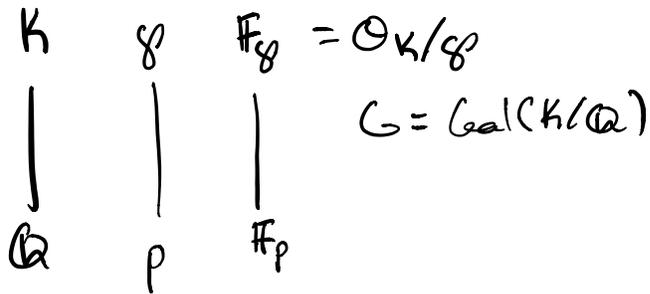
Let  $p$  be prime.

We want to make sense of an element

$$\rho(\sigma_p) \in GL_n(K)$$

$$\sigma_p = (x \mapsto x^p) \in G_{\mathbb{F}_p}$$

Recall Let  $K/\mathbb{Q}$  finite Gal,  $p$  prime.



$$D_\wp = \{\sigma \in G : \sigma(\wp) = \wp\} \leq G \quad \text{decomposition gp}$$

$$\begin{array}{c} \sigma \\ \downarrow \\ \sigma \end{array} : \mathcal{O}_K \longrightarrow \mathcal{O}_K$$

$$\begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \\ \sigma \end{array} : \mathbb{F}_\wp \longrightarrow \mathbb{F}_p$$

$$1 \longrightarrow I_\wp \longrightarrow D_\wp \longrightarrow \text{Gal}(\mathbb{F}_\wp/\mathbb{F}_p) \longrightarrow 1$$

$\uparrow$  inertia gp

$$\sigma_p \in \text{Gal}(\mathbb{F}_\wp/\mathbb{F}_p)$$

$$\updownarrow \\ \sigma_p \in D_\wp / I_\wp$$

$$\updownarrow \text{ \# unr}$$

$$\sigma_p \in D_\wp \leq G$$

$K/\mathbb{Q}$  is unramified at  $p$

$$\updownarrow$$

$$I_\wp = 1$$

well-defined up to conjugacy.

Want to have absolute decomposition + inertia gps  
 $D_p$  ,  $I_p \leq G_a$

$$1 \rightarrow I_p \rightarrow D_p \rightarrow \text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \rightarrow 1$$

$\cup$   
 $\tau_p$  topological generator

Suppose that we've defined  $D_p, I_p$

Defn Let  $\rho: G_a \rightarrow \text{GL}_n(K)$  Gal rep,

$p$  a prime number,

$D_p \leq G_a$  be a choice of decomposition gp.

$\rho$  is unramified at  $p$  if

$$I_p \subseteq \ker \rho.$$

If  $\rho$  is unram'd at  $p$ , then any preimage of  $\tau_p$  under  $D_p \rightarrow \text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$  gets sent to the same thing, so we get an elt  $\rho(\tau_p) \in \text{GL}_n(K)$ .

Different choices of  $D_p$  give conjugates of  $\rho(\tau_p)$ .

## Defn of $D_p$

$$\begin{array}{ccc} \bar{\mathbb{Q}} & \wp & \bar{\mathbb{F}}_p = \mathcal{O}_{\bar{\mathbb{Q}}} / \wp \\ | & | & | \\ \mathbb{Q} & p & \mathbb{F}_p \end{array}$$

$$D_p = \{ \sigma \in G_{\bar{\mathbb{Q}}} : \sigma(\wp) = \wp \}$$

$$1 \longrightarrow I_p \longrightarrow D_p \longrightarrow \text{Gal}(\bar{\mathbb{F}}_p / \mathbb{F}_p) \longrightarrow 1$$

Different primes  $\wp \implies$  conjugate groups  $D_p$ .