$$\varprojlim_{M/\mathbb{F}_p} \mathrm{Gal}(M/\mathbb{F}_p) \qquad I = \{ M/\mathbb{F}_p \}$$

$$\| \qquad\qquad \|$$

$$\varprojlim_{n \geq 1} \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \qquad I' = \{ \mathbb{F}_{p^n}/\mathbb{F}_p \}$$

$$\hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z} \overset{\text{CRT}}{=} \prod_p \mathbb{Z}_p$$

$\wedge$ = "profinite completion"

$$\hat{G} = \varprojlim_{\substack{H \trianglelefteq G \\ G/H \text{ finite}}} G/H$$

<u>Last time</u>

<u>Defn</u> A <u>Galois representation</u> of dimension n over a (topological) field F is a continuous homomorphism

$$\rho : G_{\mathbb{Q}} \longrightarrow GL_n(F).$$

$$G_K = \mathrm{Gal}(\overline{\mathbb{Q}}/K) \leq G_{\mathbb{Q}} \qquad K/\mathbb{Q} \text{ fin.}$$

"Galois rep of G_K" = cont. hom. $\rho : G_K \longrightarrow GL_n(\mathbb{F})$.

<u>Ex.</u> (ℓ-adic cyclotomic character)

$$\begin{array}{l} \mu_n = \{ \zeta \in \overline{\mathbb{Q}}^\times : \zeta^n = 1 \} \\ \parallel \\ \langle \zeta_n \rangle \end{array}$$

$$\begin{array}{rcl} \mathrm{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q}) & \cong & (\mathbb{Z}/n\mathbb{Z})^\times \\ \sigma & \longmapsto & a \\ \parallel & & \\ (\zeta_n \mapsto \zeta_n^a) & & \end{array}$$

$$\mathrm{Gal}(\mathbb{Q}(\mu_{\ell^\infty})/\mathbb{Q}) = \varprojlim_{n \geq 1} \mathrm{Gal}(\mathbb{Q}(\mu_{\ell^n})/\mathbb{Q}) \cong \varprojlim_{n \geq 1} (\mathbb{Z}/\ell^n\mathbb{Z})^\times$$

$$\mathbb{Q}(\mu_{\ell^\infty}) = \bigcup_{n \geq 1} \mathbb{Q}(\mu_{\ell^n}) \qquad \cong \mathbb{Z}_\ell^\times$$

$$\chi_\ell : G_{\mathbb{Q}} \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}(\mu_{\ell^\infty})/\mathbb{Q}) \cong \mathbb{Z}_\ell^\times \hookrightarrow \mathbb{Q}_\ell^\times = GL_1(\mathbb{Q}_\ell)$$

$$\begin{tikzcd}
\mathbb{Q}(\mu_{\ell^\infty}) \arrow[d, dash] & \\
\mathbb{Q}(\zeta_\ell) \arrow[d, dash] \arrow[dr, dash] & \\
\mathbb{Q} \arrow[r, dash] & \mathbb{Q}(\sqrt{\ell^*})
\end{tikzcd}$$

$$\begin{array}{ccccccccc} & & & & G_{\mathbb{Q}} & & & & \\ & & \circlearrowright & & \circlearrowright & & \circlearrowright & & \\ \cdots & \xrightarrow{\times \ell} & \mu_{\ell^3} & \xrightarrow{\times \ell} & \mu_{\ell^2} & \xrightarrow{\times \ell} & \mu_\ell \\ & & \circlearrowright & & \circlearrowright & & \circlearrowright \\ \cdots & \longrightarrow & \mathbb{Z}/\ell^3\mathbb{Z} & \longrightarrow & \mathbb{Z}/\ell^2\mathbb{Z} & \longrightarrow & \mathbb{Z}/\ell\mathbb{Z} \end{array} \implies \begin{array}{c} G_{\mathbb{Q}} \\ \circlearrowright \\ \varprojlim_n \mu_{\ell^n} \\ \circlearrowright \\ \mathbb{Z}_\ell \end{array}$$

$T = \varprojlim_n \mu_{\ell^n}$ is a rank 1 $\mathbb{Z}_\ell$-module with

a compatible $G_{\mathbb{Q}}$ action

$T \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ rk 1 $\mathbb{Q}_\ell$ module w/ $G_{\mathbb{Q}}$-action

$$\Updownarrow$$

$\rho : G_{\mathbb{Q}} \longrightarrow GL(T \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)$.

"Forming a Tate module"

Suppose $\rho : G_{\mathbb{Q}} \longrightarrow GL_n(\mathbb{C})$

$G_{\mathbb{Q}}$ and $\mathbb{C}$ have very different topologies!
- $G_{\mathbb{Q}}$ has arbitrarily small subgps
- $GL_n(\mathbb{C})$ has "no small subgps"

<u>Prop</u> Let $\rho : G_{\mathbb{Q}} \longrightarrow GL_n(\mathbb{C})$ be a complex Gal rep. Then ρ factors as

$$G_{\mathbb{Q}} \twoheadrightarrow \mathrm{Gal}(K/\mathbb{Q}) \longrightarrow GL_n(\mathbb{C})$$

for some fn. Gal. $K/\mathbb{Q}$.

Pf. Let $V \subseteq GL_n(\mathbb{C})$ open nbhd of 1 s.t. any $H \leq GL_n(\mathbb{C})$ contained in V is trivial.

$$\rho^{-1}(V) \subseteq G_{\mathbb{Q}} \text{ open nbhd of } 1.$$

So $\exists K/\mathbb{Q}$ fn. Gal. extn s.t.

$$\mathrm{Gal}(\overline{\mathbb{Q}}/K) \subseteq \rho^{-1}(V).$$

$$\rho(\mathrm{Gal}(\overline{\mathbb{Q}}/K)) \subseteq V \Rightarrow \rho(\mathrm{Gal}(\overline{\mathbb{Q}}/K)) = \{1\}.$$

Hence $\mathrm{Gal}(\overline{\mathbb{Q}}/K) \leq \mathrm{Ker}\,\rho$, so

$$\rho : G_{\mathbb{Q}} \twoheadrightarrow G_{\mathbb{Q}}/G_K = \mathrm{Gal}(K/\mathbb{Q}) \longrightarrow GL_n(\mathbb{C}). \square$$

<u>Ramification</u>

$$\rho : G_{\mathbb{Q}} \longrightarrow GL_n(K)$$

Let p be prime.

We want to make sense of an element

$$\rho(\sigma_p) \in GL_n(K)$$

$$\sigma_p = (x \longmapsto x^p) \in G_{\mathbb{F}_p}$$

__Recall__ Let $K/\mathbb{Q}$ finite Gal, p prime.

$$\begin{array}{ccc} K & \mathfrak{P} & \mathbb{F}_{\mathfrak{P}} = \mathcal{O}_K/\mathfrak{P} \\ | & | & | \\ \mathbb{Q} & p & \mathbb{F}_p \end{array} \qquad G = \mathrm{Gal}(K/\mathbb{Q})$$

$$D_{\mathfrak{P}} = \{\sigma \in G : \sigma(\mathfrak{P}) = \mathfrak{P}\} \leq G \qquad \text{decomposition gp}$$

$$\begin{array}{l} \sigma : \mathcal{O}_K \longrightarrow \mathcal{O}_K \\ \qquad \rightsquigarrow \\ \bar{\sigma} : \mathbb{F}_{\mathfrak{P}} \longrightarrow \mathbb{F}_{\mathfrak{P}} \end{array}$$

$$1 \longrightarrow I_{\mathfrak{P}} \longrightarrow D_{\mathfrak{P}} \longrightarrow \mathrm{Gal}(\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_p) \longrightarrow 1$$

inertia gp

$$\sigma_p \in \mathrm{Gal}(\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_p)$$
$$\updownarrow$$
$$\sigma_p \in D_{\mathfrak{P}}/I_{\mathfrak{P}}$$
$$\updownarrow \text{ if unr}$$
$$\sigma_p \in D_{\mathfrak{P}} \leq G$$

$K/\mathbb{Q}$ is unramified at p
$$\Updownarrow$$
$$I_{\mathfrak{P}} = 1$$

well-defined up to conjugacy.

Want to have absolute decomposition + inertia gps
D_p , $I_p \leq G_{\mathbb{Q}}$

$$1 \longrightarrow I_p \longrightarrow D_p \longrightarrow \mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \longrightarrow 1$$

$$\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \ni \sigma_p \text{ topological generator}$$

Suppose that we've defined D_p, I_p

Defn Let $\rho: G_{\mathbb{Q}} \longrightarrow GL_n(K)$ Gal rep,
p a prime number,
$D_p \leq G_{\mathbb{Q}}$ be a choice of decomposition gp.
ρ is unramified at p if

$$I_p \subseteq \ker \rho.$$

If ρ is unram'd at p, then any preimage of σ_p under $D_p \longrightarrow \mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ gets sent to the same thing, so we get an elt $\rho(\sigma_p) \in GL_n(K)$.

Different choices of D_p give conjugates of $\rho(\sigma_p)$.

Defn of D_p

$$\begin{array}{ccc} \overline{\mathbb{Q}} & \mathfrak{p} & \overline{\mathbb{F}}_p = \mathcal{O}_{\overline{\mathbb{Q}}}/\mathfrak{p} \\ | & | & | \\ \mathbb{Q} & p & \mathbb{F}_p \end{array}$$

$$D_p = \{ \sigma \in G_{\mathbb{Q}} : \sigma(\mathfrak{p}) = \mathfrak{p} \}$$

$$1 \longrightarrow I_p \longrightarrow D_p \longrightarrow \mathrm{Gal}(\overline{\mathbb{F}}_p / \mathbb{F}_p) \longrightarrow 1$$

Different primes $\mathfrak{p}$ $\Longrightarrow$ conjugate groups D_p.