Lecture for Monday: functional equation
for $\zeta(s)$, L-functions

Office hours: Thursday 1 - 2 pm
Same Zoom link as class.

<u>Last time</u>

Galois L/K (not nec. finite).

$$\mathrm{Gal}(L/K) \cong \varprojlim_{\substack{L/M/K \\ \text{finite Gal}}} \mathrm{Gal}(M/K)$$

discrete topology on $\mathrm{Gal}(M/K)$
$\overset{\text{induces}}{\rightsquigarrow}$ topology on $\mathrm{Gal}(L/K)$

Want to talk about continuous reps

$G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

<u>Rmk</u> A <u>topological group</u> is a topological space G with a "compatible" group structure.

$m: G \times G \longrightarrow G$
$\iota: G \longrightarrow G$

satisfying
- $\exists e \in G$ s.t. $m(g, \iota(g)) = e$ $\forall g \in G$
- $\vdots$
- $\vdots$

"compatible" = m, ι are continuous

<u>Ex.</u> $\mathbb{R}$

$+: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$

$-: \mathbb{R} \longrightarrow \mathbb{R}$

$\mathrm{Gal}(L/K)$ is a topological group!

Later in the course, will see abelian varieties

A: projective variety w/ compatible gp structure

$m: A \times A \longrightarrow A$

$\iota: A \longrightarrow A$

m, ι are morphisms.

<u>Prop</u> Let L/K be Galois.

(1) $\mathrm{Gal}(L/K)$ is compact

(2) For all Galois subextns $L/M/K$ the restriction map

$$\mathrm{Gal}(L/K) \longrightarrow \mathrm{Gal}(M/K)$$

is continuous + surjective w/ kernel $\mathrm{Gal}(L/M)$.

(3) For all finite Gal subextns $L/M/K$, $\mathrm{Gal}(L/M) \subseteq \mathrm{Gal}(L/K)$ is normal, open, + closed.

<u>Pf.</u> (2) $\Longrightarrow$ (3) ✓

(1).

$$\mathrm{Gal}(L/K) \cong \varprojlim_M \mathrm{Gal}(M/K) \subseteq \prod_M \mathrm{Gal}(M/K)$$

each $\mathrm{Gal}(M/K)$ is compact.

By Tychonoff's Thm, $\prod_M \mathrm{Gal}(M/K)$ is compact.

$\varprojlim$ is given by relations $\varphi_{ij}(\alpha_i) = \alpha_j$

$\Longrightarrow \varprojlim \subseteq \prod$ is closed.

$\Longrightarrow \varprojlim$ is compact.

(2) Let $\Theta : \mathrm{Gal}(L/K) \longrightarrow \mathrm{Gal}(M/K)$.

Surj: Same as in finite Galois Thy.

$$\begin{tikzcd} L \arrow[r, dashed] & \overline{K} \\ M \arrow[u, "\text{alg}"] \arrow[ur] & K \arrow[l] \arrow[u] \end{tikzcd}$$

by Zorn's lemma (extn's are in bijection with $\mathrm{Gal}(L/M)$.)

By defn $\ker\Theta = \mathrm{Gal}(L/M)$.

Continuity:

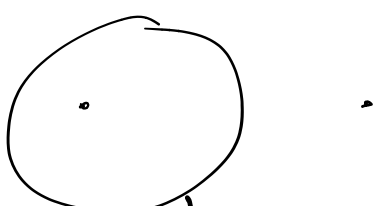
$$
\begin{tikzcd}
\mathrm{Gal}(L/K) \arrow[r, "\theta"] & \mathrm{Gal}(M/K) \\
\sigma \arrow[r, mapsto] & \sigma|_M \\
\varprojlim_{L/N/K} \mathrm{Gal}(N/K) \arrow[u, phantom, "\cong"] \arrow[r] \arrow[d, hook] & \varprojlim_{M/N/K} \mathrm{Gal}(N/K) \arrow[u, phantom, "\cong"] \arrow[d, hook] \\
\prod_{L/N/K} \mathrm{Gal}(N/K) \arrow[r, "\text{proj}"] & \prod_{M/N/K} \mathrm{Gal}(N/K)
\end{tikzcd}
$$

$$(\sigma_N)_N \longmapsto (\sigma|_M|_N)_N = (\sigma_N)_N$$

continuous

so θ is continuous. $\square$

<u>Facts</u> $\mathrm{Gal}(L/K)$ is totally disconnected.



clopen

translate of $\underline{\mathrm{Gal}(L/M)}$ $L/M/K$ finite Gal.

$\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$.

- compact,
- totally disconnected.

Thm Let L/K be Galois with $G = \mathrm{Gal}(L/K)$.
There is an inclusion-reversing bijection

$$\left\{\begin{matrix}\text{closed subgps}\\ H \le G\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{subextns}\\ L/M/K\end{matrix}\right\}$$

$$H \longmapsto L^H$$

$$\mathrm{Gal}(L/M) \longleftarrow\!\shortmid\ M$$

which restricts to bijections

$$\left\{\begin{matrix}\text{normal closed subgps}\\ H \le G\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Galois subextns}\\ L/M/K\end{matrix}\right\}$$

$$\left\{\begin{matrix}\text{open subgps}\\ H \le G\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{finite subextns}\\ L/M/K\end{matrix}\right\}.$$

<u>Pf.</u>

<u>Step 1.</u>

If $L/M/K$ is $\left\{\begin{matrix}\checkmark\ \text{finite}\\ \checkmark\ \text{Galois}\\ \text{arbitrary}\end{matrix}\right\}$ then $\mathrm{Gal}(L/M)$ is $\left\{\begin{matrix}\checkmark\ \text{open}\\ \checkmark\ \text{normal}\\ \text{closed}\end{matrix}\right\}$

$L/M/K$ finite: $\exists\, L/N/M$ s.t. N/K is fin. Gal.

$$\mathrm{Gal}(L/M) \supseteq \overbrace{\mathrm{Gal}(L/N)}^{\text{open}}$$

$$\Rightarrow \mathrm{Gal}(L/M) = \bigcup_{H \in \mathrm{Gal}(L/M)/\mathrm{Gal}(L/N)} H \quad \text{is open.}$$

$L/M/K$ Galois: By prop, $\mathrm{Gal}(L/M)$ is normal.

$L/M/K$ any subextn:

$$M = \bigcup_{\substack{M/M_i/K \\ \text{finite Gal.}}} M_i \qquad \text{each } \mathrm{Gal}(L/M_i) \text{ is closed}$$

$$\mathrm{Gal}(L/M) = \bigcap_{M_i} \mathrm{Gal}(L/M_i) \quad \text{is closed.}$$

It is clear that

$$H \longmapsto L^H$$
$$\mathrm{Gal}(L/M) \longleftarrow\!\shortmid M$$

are inclusion-reversing.

<u>Step 2</u>. They are inverses.

$L/M/K$ a subextn.

$$L^{\mathrm{Gal}(L/M)} = M \quad \text{b/c} \quad L/M \text{ is Galois.}$$

$H \leq G$ closed.

$$H \leq \mathrm{Gal}(L/L^H)$$

suffices to show H is dense in $\mathrm{Gal}(L/L^H)$.

Let $g \in \mathrm{Gal}(L/L^H)$. Any neighborhood of g

contains some $g \cdot \mathrm{Gal}(L/M)$ for some finite Galois $L/M/K$. Let

$$\begin{array}{ccc} \mathrm{Gal}(L/K) & \overset{\theta}{\twoheadrightarrow} & \mathrm{Gal}(M/K) \\ \rotatebox{90}{$\le$} & & \rotatebox{90}{$\le$} \\ H & \longrightarrow & \theta(H) \end{array}$$

Since g fixes L^H, g also fixes

$$M \cap L^H = M^{\theta(H)}$$

$$\xRightarrow{\text{fn. Gal Thy}} \theta(g) \in \mathrm{Gal}(M/M^{\theta(H)}) = \theta(H)$$

$$\Longrightarrow \exists h \in H \text{ s.t. } \theta(h) = \theta(g).$$

$$\Longrightarrow g^{-1}h \in \mathrm{Gal}(L/M)$$

$$\Longrightarrow h \in g\,\mathrm{Gal}(L/M)$$

So $H \cap g\,\mathrm{Gal}(L/M) \neq \emptyset$.

Step 3

If $H \leq G$ is $\left\{ \begin{array}{c} \text{open} \\ \text{normal} \\ \text{+ closed} \end{array} \right\}$ then L^H/K is $\left\{ \begin{array}{c} \text{finite} \\ \text{Galois} \end{array} \right\}$.

$H \leq G$ open: $H \supseteq \mathrm{Gal}(L/M)$ for some fn. Gal. $L/M/K$.

$$\Downarrow$$

$$L^H \subseteq M \Longrightarrow L^H/K \text{ is finite.}$$

$H \leq G$ normal & closed: $\forall\, L/M/K$ finite Gal

$M^{\sigma(H)}/K$ is Gal (by fn. Gal. Thy)

$L^H = \bigcup_M M^H$ is Galois too. $\square$

<u>Galois Representations</u>

$G_{\mathbb{Q}}$ is topolgical group.

$\rho : G_{\mathbb{Q}} \longrightarrow GL_n(\mathbb{C})$ n-dimensional rep.

impose that ρ is continuous

$GL_n(\mathbb{C})$ by $GL_n(K)$ for $K = \mathbb{Q}_\ell, \ldots$

K can be a $\boxed{\text{topological field.}}$

$K = \mathbb{C}, \mathbb{Q}_\ell, K/\mathbb{Q}_\ell, \mathbb{F}_\ell$ ← discrete.

<u>Defn</u> A <u>Galois representation</u> of dimension n over a (topological) field K is a continuous homomorphism

$$\rho : G_{\mathbb{Q}} \longrightarrow GL_n(K).$$

Ex. ℓ-adic cyclotomic character.

ℓ : prime number

$$\mu_n = \{ \zeta \in \overline{\mathbb{Q}} : \zeta^n = 1 \}$$

$$\mathrm{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times .$$

$$\sigma \longmapsto a \quad \text{s.t.} \quad \sigma(\zeta_n) = \zeta_n^a .$$

$$\mu_{\ell^\infty} = \bigcup_{n \geq 1} \mu_{\ell^n} .$$

$$\begin{aligned} \mathrm{Gal}(\mathbb{Q}(\mu_{\ell^\infty})/\mathbb{Q}) &\cong \varprojlim_{n \geq 1} \mathrm{Gal}(\mathbb{Q}(\mu_{\ell^n})/\mathbb{Q}) \\ &\cong \varprojlim_{n \geq 1} (\mathbb{Z}/\ell^n\mathbb{Z})^\times \\ &\cong \mathbb{Z}_\ell^\times \end{aligned}$$

$$\chi_\ell : G_\mathbb{Q} \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}(\mu_{\ell^\infty})/\mathbb{Q}) \cong \mathbb{Z}_\ell^\times \hookrightarrow \mathbb{Q}_\ell^\times$$

$$= GL_1(\mathbb{Q}_\ell)$$