Last time

Constructed Hecke operators $\{T_p\}_{p\nmid N}$ on $\mathcal{M}_{2k}(\Gamma_0(N))$

s.t.

$$T_p(f) = p^{k-1}\left(\sum_{j=0}^{p-1} f\Big|\begin{pmatrix}1 & j\\ 0 & p\end{pmatrix} + f\Big|\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}\right)$$

$$= \frac{1}{p}\sum_{j=0}^{p-1} f\left(\frac{\tau+j}{p}\right) + p^{2k-1} f(p\tau)$$

T_p preserves $S_{2k}(\Gamma_0(N))$.
$T_pT_q = T_qT_p \quad \forall p,q\nmid N$. (check using the formula!)

Moreover, if $f = q + \sum_{n\geq 2} a_n q^n \in S_{2k}(\Gamma_0(N))$ is

a normalized eigenform for T_p then

$$T_pf = a_p f.$$

Today

- Define T_n for $(n, N) = 1$

- Explain why each T_n can be diagonalized.

Since T_n commute with each other, they are simultaneously diagonalizable!

I.e. there is a basis $\{f_i\}$ for $S_{2k}(\Gamma_0(N))$ s.t. each f_i is an eigenform for all T_n $(n,N)=1$.

<u>Defn of T_n</u> $T_1 = 1$.

Defining T_{p^r} inductively by

$$T_{p^r} = T_p T_{p^{r-1}} - p^{2k-1} T_{p^{r-2}} .$$

Then if $n = \prod_{p \nmid N} p^{e_p}$ then define

$$T_n = \prod_{p \nmid N} T_{p^{e_p}} .$$

All T_n still commute and if $(m,n)=1$ then $T_m T_n = T_{mn}$.

Why the strange defn?

$$\sum_{\substack{(n,N)=1 \\ n \geq 1}} T_n n^{-s} = \prod_{p \nmid N} \frac{1}{1 - T_p p^{-s} + p^{2k-1-2s}}$$

This looks like some quadratic poly $1 - T_p X + p^{2k-1} T^2$ evaluated at $T = p^{-s}$.

Petersson inner product

$S_{2k}(\Gamma_0(N))$ has a Hermitian inner product $\langle \cdot, \cdot \rangle$.

Prop T_n is self-adjoint under $\langle \cdot, \cdot \rangle$, i.e.

$$\langle T_n f, g \rangle = \langle f, T_n g \rangle \quad \forall f, g \in S_{2k}(\Gamma_0(N)).$$

Thm (Spectral thm for self-adjoint operators).
If A is a self-adjoint operator on a fin.-dim'l Hermitian inner product space V, then there is an orthonormal basis for V consisting of eigenvectors for A.

Defn The *hyperbolic measure* on $\mathcal{H}$ is

$$d\mu(\tau) = \frac{dx\,dy}{y^2} \quad \text{where} \quad \tau = x + iy.$$

Prop $d\mu(\tau)$ is invariant under $GL_2^+(\mathbb{Q})$.
I.e. $d\mu(\gamma\tau) = d\mu(\tau)$ $\forall \gamma \in GL_2^+(\mathbb{Q})$.

<u>Note</u> If $\mathcal{F}$ is the standard fund domain for $SL_2(\mathbb{Z})$ then

$$\mathrm{vol}(\mathcal{F}) = \int_{\mathcal{F}} 1 \cdot d\mu(\tau) \leq \int_{-1/2}^{1/2} \left(\int_{\sqrt{3}/2}^{\infty} \frac{dy}{y^2} \right) dx < \infty .$$

More generally, if f is a bounded fn on $\mathcal{F}$

then $\left| \int_{\mathcal{F}} f(\tau) d\mu(\tau) \right| < \infty$.

If $\Gamma \leq SL_2(\mathbb{Z})$ is a cong subgp with fund. domain

$D = \bigcup_{\gamma_i} \gamma_i \mathcal{F}$ where $\{\gamma_i\}$ is a set of reps for $\Gamma \backslash SL_2(\mathbb{Z})$

we will write

$$\int_{X(\Gamma)} d\mu(\tau) \text{ for } \int_D d\mu(\tau) .$$

If f is a Γ-invariant function $(f(\gamma\tau) = f(\tau) \ \forall \gamma \in \Gamma)$

then $\int_{X(\Gamma)} f(\tau) d\mu(\tau) = \int_D f(\tau) d\mu(\tau)$ does not depend on the choice of reps $\{\gamma_i\}$

__Note__ $\mathrm{vol}(X(\Gamma)) = \int_{X(\Gamma)} d\mu(\tau) = [SL_2(\mathbb{Z}) : \{\pm 1\}\Gamma]\, \mathrm{vol}(X(SL_2(\mathbb{Z})))$.

__Warning__ If $f \in M_{2k}(\Gamma)$ then $\int_{X(\Gamma)} f(\tau)\, d\mu(\tau)$

does __not__ make sense because f is __not__ Γ-invariant.

__But__ if $f, g \in M_{2k}(\Gamma)$, then the function

$f(\tau)\,\overline{g(\tau)}\, \mathrm{Im}(\tau)^{2k}$ __is__ Γ-invariant:

$$f(\gamma\tau)\,\overline{g(\gamma\tau)}\, \mathrm{Im}(\gamma\tau)^{2k} = (c\tau+d)^k f(\tau)\, \overline{(c\tau+d)}^k\, \overline{g(\tau)}\, \frac{\mathrm{Im}(\tau)^{2k}}{|c\tau+d|^{2k}}$$

$$= f(\tau)\,\overline{g(\tau)}\, \mathrm{Im}(\tau)^{2k}.$$

So the integral $\int_{X(\Gamma)} f(\tau)\,\overline{g(\tau)}\, \mathrm{Im}(\tau)^{2k}\, d\mu(\tau)$

makes sense but does not necessarily converge!

$\mathrm{Im}(\tau)$ is unbounded on $\mathcal{H}$.

To ensure convergence, restrict to cusp forms:

If $f \in S_{2k}(\Gamma)$ then $f(\tau) = \sum_{n\geq 1} a_n q_h^n$

where $q_h = e^{2\pi i \tau/h}$ with $h \geq 1$, we see

$|f(\tau)|$ decays exponentially as $\mathrm{Im}(\tau) \to \infty$.

Hence $\int_{X(\Gamma)} f(\tau)\overline{g(\tau)}\,\mathrm{Im}(\tau)^{2k}\,d\mu(\tau)$ converges!

(Note that we only needed f or $g \in S_{2k}$, not both)

<u>**Defn**</u> The Petersson inner product on $S_{2k}(\Gamma)$ is defined by

$$\langle f, g\rangle_\Gamma = \frac{1}{\mathrm{vol}(X(\Gamma))}\int_{X(\Gamma)} f(\tau)\,\overline{g(\tau)}\,\mathrm{Im}(\tau)^{2k}\,d\mu(\tau).$$

<u>**Obs**</u> $\langle\cdot,\cdot\rangle_\Gamma$ is ...

(1) linear in f
(2) conjugate linear in g
(3) Hermitian i.e. $\langle f,g\rangle = \overline{\langle g,f\rangle}$
(4) positive definite i.e. $\langle f,f\rangle = 0 \iff f = 0$

Also if $\Gamma' \subseteq \Gamma$ and $f, g \in S_{2k}(\Gamma) \subseteq S_{2k}(\Gamma')$ then the normalization ensures

$$\langle f, g\rangle_{\Gamma} = \langle f, g\rangle_{\Gamma'}.$$

<u>Prop</u> T_n is self-adjoint for $\langle \cdot, \cdot \rangle_{\Gamma_0(N)}$ where $(n, N) = 1$.

<u>Pf.</u> (sketch) Suffices to show for T_p, $p \nmid N$ prime.
Use the formula

$$T_p f = p^{k-1}\left(\sum_{j=0}^{p-1} f\Big|\begin{pmatrix}1 & j\\ 0 & p\end{pmatrix} + f\Big|\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}\right)$$

and the fact that f, g are modular (and some algebra) to compute.
(Computation can be found in Diamond + Shurman §5.5.) $\square$

<u>Corollary</u> $S_{2k}(\Gamma_0(N))$ has an orthonormal basis $\{f_i\}$ where each f_i is a normalized eigenform for all T_n. In particular $T_p f = a_p f$ where $f = q + \sum_{n \geq 2} a_n q^n$.

Using geometric techniques, we get some deeper properties!

Let $\mathbb{T} = \mathbb{Z}[T_n : n \geq 1 \ (n,N)=1]$ be the Hecke algebra.
$= \mathbb{Z}[T_p : p \nmid N]$

Fun fact $\mathbb{T}$ is finitely generated!

Fun fact The characteristic polynomial of T_p on $S_{2k}(\Gamma)$ has integral coefficients!

Consequence If f is a normalized Hecke eigenform for T_p then

$$0 = h(T_p)f = h(a_p)f \implies h(a_p) = 0.$$

h = char poly of $T_p \in \mathbb{Z}[X]$

So a_p is an algebraic integer!

Consequence If f is a normalized eigenform for all T_p, then we have a map

$$\lambda : \mathbb{T} \longrightarrow \mathbb{C}$$
$$T_p \longmapsto a_p.$$

Since $\mathbb{T}$ is fin-gen'd, so is the $\mathbb{Z}$-module $\mathrm{im}(\lambda) \subseteq \mathbb{C}$.
By the above, $\mathbb{Q}(a_n : n \geq 1)$ is a number field!
(i.e. fin extention of $\mathbb{Q}$.)

Let $f = \sum_{n \geq 1} a_n q^n \in S_{2k}(SL_2(\mathbb{Z}))$ be a normalized eigenform for all T_n.

$$L(f, s) = \sum_{n \geq 1} a_n n^{-s} = \prod_{p < \infty} \frac{1}{1 - a_p p^{-s} + p^{2k-1-2s}}$$

$$g(s) = \int_0^\infty f(it)\, t^s \frac{dt}{t} = \sum_{n \geq 1} a_n \int e^{-2\pi t} t^{s-1} dt$$

Mellin transform

$$= (2\pi)^{-s} \Gamma(s) \sum_{n \geq 1} a_n n^{-s}$$

$$= (2\pi)^{-s} \Gamma(s) L(f, s)$$