Announcements

- Final paper due 8/31.
- Tell me your topic choice by Monday.

Today

Hecke operators.

Recall $\Delta \in S_{12}(SL_2(\mathbb{Z}))$, $\Delta(\tau) = \sum_{n \geq 1} \tau(n) q^n$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\tau(n)$	1	−24	252	−1472	4830	−6048	−16744	84480	−113643	−115920	534612	−370944	−577738	401856	1217160	987136

$$\tau(2)\cdot\tau(3) = -6048 = \tau(6)$$
$$\tau(2)\cdot\tau(5) = -115920 = \tau(10)$$
$$\tau(2)\cdot\tau(2) = 576 \neq -1472 = \tau(4)$$
$$\tau(4) - \tau(2)\tau(2) = -2048 = -2^{11}$$
$$\tau(8) - \tau(2)^3 = 98304 = 2^{15}\cdot 3 = 2^{12}\cdot 24$$
$$\tau(8) - \tau(2)\tau(4) = 49152 = 2^{14}\cdot 3 = 2^{11}\cdot 24$$
$$\tau(4)\tau(3) = \tau(12) \checkmark$$

Conjecture: if $(m,n)=1$ then $\tau(m)\tau(n) = \tau(mn)$.

Q: Does this hold for more mod forms?
If so, which ones?

We have a finite dim'l vector space $S_k(\Gamma)$. Can we find a basis of this space consisting of forms with this sort of property?

Yes we can!

Hecke operators

Idea: construct a family of operators $\{T_p\}_{p \nmid N \text{ prime}}$ on $S_k(\Gamma_0(N))$. Each T_p is diagonalizable, and they commute among each other

$\Longrightarrow$ there is a basis for $S_k(\Gamma_0(N))$ consisting of simultaneous <u>eigenforms</u>

These eigenforms will have the properties from above, e.g. if $\sum a_n q^n$ is an eigenform then $a_m a_n = a_{mn}$ when $(m,n)=1$.

<u>Recall</u>

$$S_0(N) = \left\{ (E,C) : \begin{array}{c} E \text{ ell. curve} \\ C \subseteq E[N] \text{ cyclic} \\ \text{of order } N \end{array} \right\} \Big/ \text{iso.} \longleftrightarrow \begin{array}{c} Y(\Gamma_0(N)) \\ \| \\ Y_0(N). \end{array}$$

If $p \nmid N$, note that a cyclic subgp $C \leq E[pN]$ decomposes uniquely as

$$C = C_N \oplus C_p \qquad \begin{array}{l} C_N \text{ cyclic of order } N \\ C_p \text{ cyclic of order } p. \end{array}$$

Thus, we have two maps

$$S_0(pN) \longrightarrow S_0(N)$$

$$\pi_1 : (E, C) \longmapsto (E, C_N)$$

$$\pi_2 : (E, C) \longmapsto (E/C_p, C/C_p)$$

gives:

$$\begin{tikzcd}
 & Y_0(pN) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
Y_0(N) & & Y_0(N)
\end{tikzcd}$$

From a situation like this, we can obtain maps

$$\mathrm{Div}(Y_0(N)) \xrightarrow{\pi_1^*} \mathrm{Div}(Y_0(pN)) \xrightarrow{(\pi_2)_*} \mathrm{Div}(Y_0(N))$$

$$(\tau) \longmapsto (\pi_2(\tau))$$

$$\Omega^{\otimes k}(Y_0(N)) \xrightarrow{\pi_1^*} \Omega^{\otimes k}(Y_0(pN)) \xrightarrow{(\pi_2)_*} \Omega^{\otimes k}(Y_0(N))$$

what does this mean?

The map constructed this way is called a <u>Hecke operator</u> (T_p)

Let's study π_1, π_2 more closely.

$$\begin{tikzcd}
S_0(pN) \arrow[rrr, "\pi_1"] & & & S_0(N) \\
& (\mathbb{C}/\Lambda_\tau, \langle \tfrac{1}{pN} \rangle) \arrow[r, mapsto] & (\mathbb{C}/\Lambda_\tau, \langle \tfrac{1}{N} \rangle) & \\
& \tau \arrow[u, mapsto] \arrow[r, mapsto] & \tau \arrow[u, mapsto] & \\
Y_0(pN) \arrow[uuu] \arrow[rrr, "\pi_1"'] & & & Y_0(N) \arrow[uuu]
\end{tikzcd}$$

so $\pi_1 : \Gamma_0(pN)\backslash\mathcal{H} \longrightarrow \Gamma_0(N)\backslash\mathcal{H}$

is the natural quotient map.

$$\begin{tikzcd}
S_0(pN) \arrow[rrr, "\pi_2"] & & & S_0(N) \\
& (\mathbb{C}/\Lambda_\tau, \langle \tfrac{1}{pN} \rangle) \arrow[r, mapsto] & (\mathbb{C}/_{\tau\mathbb{Z}+\frac{1}{p}\mathbb{Z}}, \langle \tfrac{1}{pN} \rangle) & \\
& & (\mathbb{C}/_{p\tau\mathbb{Z}+1\mathbb{Z}}, \langle \tfrac{1}{N} \rangle) \arrow[u, phantom, "\shortparallel"] & \\
& \tau \arrow[uu, mapsto] \arrow[r, mapsto] & p\tau \arrow[u, mapsto] & \\
Y_0(pN) \arrow[uuuu] \arrow[rrr, "\pi_2"'] & & & Y_0(N) \arrow[uuuu]
\end{tikzcd}$$

For $\alpha_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$, this factors as

$$\begin{tikzcd}
\Gamma_0(pN)\backslash\mathcal{H} \arrow[rr, "\alpha_p"] \arrow[ddrr, "\pi_2"'] & & \alpha_p\Gamma_0(pN)\alpha_p^{-1}\backslash\mathcal{H} \arrow[dd, "q"] \\
\tau \arrow[r, mapsto] \arrow[dr, mapsto] & \alpha_p\tau = p\tau \arrow[d, mapsto] & \\
& p\tau & \\
& & \Gamma_0(N)\backslash\mathcal{H}
\end{tikzcd}$$

Claim:

(1) If $\Gamma \leq SL_2(\mathbb{Z})$, $\alpha \in SL_2(\mathbb{Z})$ then

$$\begin{aligned} \Gamma\backslash\mathcal{H} &\longrightarrow \alpha\Gamma\alpha^{-1}\backslash\mathcal{H} \\ \tau &\longmapsto \alpha\tau \end{aligned}$$

is an isomorphism.

(2) $\alpha_p\Gamma_0(pN)\alpha_p^{-1} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \ : \ N \mid c,\ p \mid b \right\} \leq \Gamma_0(N)$

Pf.

(1) Well-defined: if $\tau' = \gamma\tau$ some $\gamma \in \Gamma$, then

$$\alpha\tau' = (\alpha\gamma\alpha^{-1})(\alpha\tau).$$

Inverse is given by

$$\begin{aligned} \alpha\Gamma\alpha^{-1}\backslash\mathcal{H} &\longrightarrow \Gamma\backslash\mathcal{H} \\ \tau &\longmapsto \alpha^{-1}\tau \end{aligned}.$$

(2)

$$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1/p & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & bp \\ c/p & d \end{pmatrix} \qquad \square$$

So we have the diagram, for $\Gamma = \alpha_p \Gamma_0(pN)\alpha_p^{-1}$

$$\begin{tikzcd} \Gamma_0(pN)\backslash\mathcal{H} \arrow[r, "\alpha_p", "\sim"'] \arrow[d, "\pi_1"'] & \Gamma\backslash\mathcal{H} \arrow[d, "q"] \\ \Gamma_0(N)\backslash\mathcal{H} & \Gamma_0(N)\backslash\mathcal{H} \end{tikzcd}$$

Define a map $\Omega^{\otimes k}(Y_0(N)) \longrightarrow \Omega^{\otimes k}(Y_0(N))$ as:

$$\Omega^{\otimes k}(Y_0(N)) \xrightarrow{\pi^*} \Omega^{\otimes k}(Y_0(pN)) \xrightarrow{(\alpha_p^{-1})^*} \Omega^{\otimes k}(Y(\Gamma)) \xrightarrow{q_*} \Omega^{\otimes}(Y_0(N))$$

Trace

$$\omega \longmapsto \sum_{\gamma \in R} \gamma^* \omega$$

where R is a set of coset reps for $\Gamma \backslash \Gamma_0(N)$.

To compute this explicitly, need coset reps.

<u>Prop</u> A set of coset reps for $\Gamma \backslash \Gamma_0(N)$ is

$\{\gamma_j\}_{j=0,1,\dots,p-1,\infty}$ where $\gamma_j = \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}$ for $0 \le j < p$

and $\gamma_\infty = \begin{pmatrix} xp & y \\ N & 1 \end{pmatrix}$ where $xp - yN = 1$.

<u>Pf.</u> To see the represent every coset:

let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, want some $\gamma\gamma_j^{-1} \in \Gamma$

some j. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -j \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} * & b-aj \\ * & * \end{pmatrix}$

if $p \nmid a$, then take $j \equiv ba^{-1} \pmod p$.

If $p \mid a$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -y \\ -N & xp \end{pmatrix} = \begin{pmatrix} * & -ay+bxp \\ * & * \end{pmatrix} \in \Gamma$.

To see that γ_j defined distinct coset reps:

$$\gamma_{j_1} \gamma_{j_2}^{-1} \in \Gamma \iff j_1 = j_2$$

if $j_1, j_2 < \infty$, $\gamma_{j_1} \gamma_{j_2}^{-1} = \begin{pmatrix} 1 & j_1 - j_2 \\ 0 & 1 \end{pmatrix} \in \Gamma \iff j_1 = j_2$.

if $j_1 = \infty$

$$\begin{pmatrix} xp & y \\ N & 1 \end{pmatrix} \begin{pmatrix} 1 & -j_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} * & y - xpj_2 \\ * & * \end{pmatrix} \notin \Gamma.$$

$\square$

Now let's compute! Recall that

$$\begin{array}{ccc} \Omega^{\otimes k}(Y(\Gamma)) & \xrightarrow{\gamma^*} & \Omega^{\otimes k}(Y(\Gamma')) \\ \downarrow \wr & & \downarrow \wr \\ A_{2k}(\Gamma) & \xrightarrow{f \mapsto f|_\gamma} & A_{2k}(\Gamma') \end{array}$$

commutes.

$$\begin{tikzcd}
\Omega^{\otimes}(Y_0(N)) \arrow[rrr, "\pi_1^*"] \arrow[ddd] & & & \Omega^{\otimes k}(Y_0(pN)) \arrow[ddd] \\
& f(\tau)d\tau^k \arrow[r, mapsto] \arrow[d, mapsto] & f(\tau)d\tau^k \arrow[d, mapsto] & \\
& f \arrow[r, mapsto] & f & \\
A_{2k}(\Gamma_0(N)) \arrow[rrr] & & & A_{2k}(\Gamma_0(pN))
\end{tikzcd}$$

so $A_{2k}(\Gamma_0(N)) \longrightarrow A_{2k}(\Gamma_0(pN))$

is the natural inclusion.

$$\begin{tikzcd}
\Omega^{\otimes}(Y_0(pN)) \arrow[rr, "(\alpha_p^{-1})^*"] \arrow[ddd] & & \Omega^{\otimes k}(Y(\Gamma)) \arrow[rr, "q_*"] & & \Omega^{\otimes k}(Y_0(N)) \arrow[ddd] \\
& f(\tau)d\tau^k \arrow[r, mapsto] \arrow[dd, mapsto] & (f|\alpha_p^{-1})(\tau)\,d\tau^k \arrow[r, mapsto] & \sum_{\gamma_j}(f|\alpha_p^{-1}\gamma_j)(\tau)d\tau^k \arrow[dd, mapsto] & \\
& & & & \\
& f \arrow[rr, mapsto] & & \sum_{\gamma_j} f|\alpha_p^{-1}\gamma_j & \\
A_{2k}(\Gamma_0(pN)) \arrow[rrrr] & & & & A_{2k}(\Gamma_0(N))
\end{tikzcd}$$

So our composite map $A_{2k}(\Gamma_0(N)) \longrightarrow A_{2k}(\Gamma_0(N))$

is $\displaystyle f \longmapsto \sum_{\gamma_j} f|\alpha_p^{-1}\gamma_j$.

Since scaling a matrix doesn't change how it acts on a modular form, we'll replace $\alpha_p^{-1} = \begin{pmatrix} 1/p & 0 \\ 0 & 1 \end{pmatrix}$. with $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$.

Then our map is

$$f \longmapsto \sum_{j=0}^{p-1} f\Big|\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} + f\Big|\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} xp & y \\ N & 1 \end{pmatrix}$$

$$= \sum_{j=0}^{p-1} f|\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} + f\Big|\begin{pmatrix} x & y \\ N & p \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \sum_{j=0}^{p-1} f|\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} + f|\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}.$$

$$f(\tau) \longmapsto \sum_{j=0}^{p-1} p^{-k} f\left(\frac{\tau+j}{p}\right) + p^k f(p\tau).$$

If f is holomorphic on $\mathcal{H}$, then this is as well.

If $f(\tau) = \sum_{n\geq 0} a_n q^n$ then

$$f \longmapsto \sum_{j=0}^{p-1} p^{-k} \sum_{n\geq 0} a_n q^{n/p} \zeta_p^{jn} + p^k \sum_{n\geq 0} a_n q^{pn}$$

$$= p^{-k} \sum_{n\geq 0} a_n q^{n/p} \sum_{j=0}^{p-1} \zeta_p^{jn} + p^k \sum_{n\geq 0} a_n q^{pn}$$

$$= p^{-k+1} \sum_{n\geq 0} a_{pn} q^n + p^k \sum_{n\geq 0} a_n q^{pn}$$

$$= \sum_{n\geq 0} \left(p^{-k+1} a_{pn} + p^k a_{n/p}\right) q^n \quad \text{where } a_{n/p} = 0 \text{ if } p \nmid n.$$

Let T_p be this map scaled by p^{k-1}.

$$(T_p f)(\tau) = \sum_{n \geq 0} \left(a_{pn} + p^{2k-1} a_{n/p}\right) q^n .$$

Thus, if $f \in \mathcal{M}_{2k}$ then $T_p f \in \mathcal{M}_{2k}$

if $f \in S_{2k}$ then $T_p f \in S_{2k}$.

If f is an eigenform for T_p, and

$f = \sum_{n \geq 1} a_n q^n$ with $a_1 = 1$, then

$T_p f = \lambda_p f$. What is λ_p?

$\lambda_p = a_1$ term of $T_p f$

$$= a_p + p^{2k-1} a_{1/p} = a_p$$

So if f is an normalized eigenform for T_p
then $T_p f = a_p f$.

<u>Prop</u> $T_p T_q = T_q T_p$ for primes p, q.

<u>**Pf.**</u> Apply formula for $T_p f$. $\square$