Last time

- examples of modular forms: Eisenstein series

$$y^2 = 4x^3 - \underbrace{g_2(\tau)}_{\text{wt } 4} x - \underbrace{g_3(\tau)}_{\text{wt } 6}$$

- Proved: for $\Gamma \leq SL_2(\mathbb{Z})$ cong. subgp, $k \in \mathbb{Z}$

Prop There is an isomorphism of $\mathbb{C}$-vector spaces

$$\begin{aligned} A_{2k}(\Gamma) &\longrightarrow \Omega^{\otimes k}(X(\Gamma)) \\ f &\longmapsto \underbrace{\text{"} f(\tau)(d\tau)^k \text{"}}_{\omega_f} \end{aligned}$$

such that

$\mathrm{ord}_\tau(f) = h_\tau \, \mathrm{ord}_{\pi(\tau)}(\omega_f) + k(h_\tau - 1)$ if $\tau \in \mathcal{H}$ with period h_τ

$\mathrm{ord}_s(f) = \mathrm{ord}_{\pi(s)}(\omega_f) + k$ if $s \in \mathbb{P}^1(\mathbb{Q})$.

Today

- Dimension formulas
- Applications of modular forms
- Examples of mod forms.

Want to compute $\dim_{\mathbb{C}} M_{2k}(\Gamma)$

if $f \in A_{2k}(\Gamma)$ then

$$f \in \mathcal{M}_{2k}(\Gamma) \iff \begin{array}{ll} \mathrm{ord}_\tau(f) \geq 0 & \forall \tau \in \mathcal{H} \\ \mathrm{ord}_s(f) \geq 0 & \forall s \in \mathbb{P}^1(\mathbb{Q}) \end{array}$$

$$\iff \begin{array}{l} \mathrm{ord}_{\pi(\tau)}(\omega) \geq -k\left(1-\frac{1}{h_\tau}\right) \\ \mathrm{ord}_{\pi(s)}(\omega) \geq -k \end{array}$$

$$\iff \begin{array}{l} \mathrm{ord}_{\pi(\tau)}(\omega) \geq -\left\lfloor k\left(1-\frac{1}{h_\tau}\right)\right\rfloor \\ \mathrm{ord}_{\pi(s)}(\omega) \geq -k \end{array}$$

Let $0 \neq \omega_0 \in \Omega^{\otimes k}(X(\Gamma))$, and define

$$D = \mathrm{div}(\omega_0) + \sum_{\tau \in Y(\Gamma)} \left\lfloor k\left(1-\frac{1}{h_\tau}\right)\right\rfloor (\tau) + \sum_{\substack{s \text{ cusp} \\ \text{of } X(\Gamma)}} k\,(s).$$

Note that $\mathrm{div}(\omega_0) \in \mathrm{Pic}(X(\Gamma))$ is independent of the choice of ω_0, since $\dim_{\mathbb{C}(X(\Gamma))} \Omega^{\otimes k}(X(\Gamma)) = 1$

$\dim_{\mathbb{C}(X)} \Omega^1(X) = 1$

<u>Prop</u> There is an isomorphism

$$\begin{array}{ccc} \mathcal{M}_{2k}(\Gamma) & \longrightarrow & \mathcal{L}(D) = \{f \in \mathbb{C}(X(\Gamma)) : \mathrm{div}(f) \geq -D\} \\ f & \longmapsto & \dfrac{f(\tau)(d\tau)^k}{\omega_0}. \end{array}$$

Pf.

Let $\omega = f(\tau)(d\tau)^k \in \Omega^{\otimes k}(X(\Gamma))$. Then $\omega = h\,\omega_0$ some $h \in \mathbb{C}(X(\Gamma))$ where $h = \frac{f(\tau)(d\tau)^k}{\omega_0}$ in the notation of the prop.

$$f \in \mathcal{M}_{2k}(\Gamma) \iff \operatorname{div}(h\omega_0) \geq -\sum_{\tau} \lfloor k(1-\tfrac{1}{h_\tau}) \rfloor (\tau) + \sum_{s} k(s)$$

$$\iff \operatorname{div}(h) \geq -D \qquad \square$$

Thm Let $\Gamma \leq SL_2(\mathbb{Z})$ be a cong. subgp.

Let g be the genus of $X(\Gamma)$, $\varepsilon_2, \varepsilon_3$ be the # of pts of period 2,3, v_∞ be the # of cusps. Then,

$$\dim_{\mathbb{C}} \mathcal{M}_{2k}(\Gamma) = \begin{cases} 0 & k<0 \\ 1 & k=0 \\ (2k-1)(g-1) + \lfloor \frac{k}{2} \rfloor \varepsilon_2 + \lfloor \frac{2k}{3} \rfloor \varepsilon_3 + k v_\infty & k>0. \end{cases}$$

Pf. Need to compute $\ell(D)$.

$$\deg D = \underbrace{\deg(\operatorname{div}(\omega_0))}_{k(2g-2)} + \left\lfloor \frac{k}{2} \right\rfloor \varepsilon_2 + \left\lfloor \frac{2k}{3} \right\rfloor \varepsilon_3 + k v_\infty .$$

If $k<0$, $\deg D < 0$, so $\ell(D)=0$.

If $k=0$, $\mathcal{M}_{2k}(\Gamma)$ is the space of holo functions on the compact R.S. $X(\Gamma)$, so consists only of constant functions.

If $k>0$, $\deg D > 2g-2$, so

$$\deg(K_{X(\Gamma)} - D) < 0, \text{ hence } \ell(K_{X(\Gamma)} - D) = 0.$$

By Riemann-Roch,

$$\ell(D) - \ell(K_{X(\Gamma)} - D) = \deg D - g + 1$$

$$\ell(D) = k(2g-2) + \lfloor \tfrac{k}{2} \rfloor \varepsilon_2 + \lfloor \tfrac{2k}{3} \rfloor \varepsilon_3 + k v_\infty - g + 1$$

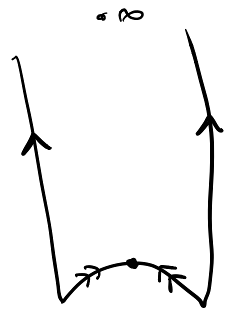
$$= (2k-1)(g-1) + \lfloor \tfrac{k}{2} \rfloor \varepsilon_2 + \lfloor \tfrac{2k}{3} \rfloor \varepsilon_3 + k v_\infty. \quad \square$$

<u>Cor</u> If $k>0$

$$\dim_{\mathbb{C}} \mathcal{M}_{2k}(SL_2(\mathbb{Z})) = -(2k-1) + \lfloor \tfrac{k}{2} \rfloor + \lfloor \tfrac{2k}{3} \rfloor + k$$

$$= \begin{cases} \lfloor \frac{2k}{12} \rfloor + 1 & 2k \not\equiv 2 \pmod{12} \\ \lfloor \frac{2k}{12} \rfloor & 2k \equiv 2 \pmod{12} \end{cases}$$

Why is $g=0$?



0	2	4	6	8	10	12	14	16	18	20	22	24
1	0	1	1	1	1	2	1	2	2	2	2	3

Applications of mod forms

Obs: If f, g are MFs of wts h, k, then $f \cdot g$ is a MF of wt. $h+k$.

$$(f\cdot g)(\gamma\tau) = f(\gamma\tau)\, g(\gamma\tau)$$
$$= (c\tau+d)^h f(\tau)\, (c\tau+d)^k g(\tau)$$
$$= (c\tau+d)^{h+k} (f\cdot g)(\tau).$$

Prop The Fourier expansion of G_k is

$$G_k(\tau) = \frac{\zeta(1-k)}{2} + \sum_{n\geq 1} \sigma_{k-1}(n)\, q^n$$

where $\sigma_{k-1}(n) = \sum_{d|n,\, d\geq 0} d^{k-1}$.

Rmk $\sigma_{k-1}(0) = \sum_{n=1}^{\infty} n^{k-1} = \zeta(1-k)$

0 is "half negative, half positive", so take $\frac{1}{2}$.

Rmk All $\sigma_{k-1}(n) \in \mathbb{Z}$. Also $\zeta(1-k) = \frac{-B_k}{k} \in \mathbb{Q}$

where B_k is the k^{th} Bernoulli #.

Notation $E_k = \frac{2}{\zeta(1-k)} G_k = 1 - \frac{2k}{B_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n$.

Application

By our dimension formulas,

$\dim_{\mathbb{C}} \mathcal{M}_8(SL_2(\mathbb{Z})) = 1$.

$E_4^2, E_8 \in \mathcal{M}_8(SL_2(\mathbb{Z}))$ both of whose q-exp's start $1 + \cdots$.

Hence $E_4^2 = E_8$!!!

Taking n^{th} Fourier coefficients:

$$-2 \cdot \frac{8}{B_4} \sigma_3(n) + \left(\frac{8}{B_4}\right)^2 \sum_{j=1}^{n-1} \sigma_3(j)\sigma_3(n-j) = \frac{-16}{B_8} \sigma_7(n)$$

Looking up B_k:

$$B_4 = B_8 = \frac{-1}{30}$$

$$\boxed{\sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(m)\, \sigma_3(n-m)}.$$

This generalizes!

On this week's exercises, do the same with $\mathcal{M}_{10}$, $\mathcal{M}_{14}$.

Further examples of modular forms

$X(SL_2(\mathbb{Z}))$ has 1 cusp. Hence

$$\dim_{\mathbb{C}} \mathcal{M}_{2k}(\Gamma) = 1 + \dim_{\mathbb{C}} S_{2k}(\Gamma)$$

if $\dim_{\mathbb{C}} \mathcal{M}_{2k}(\Gamma) \neq 0$.

So the first time we have a cusp form is in wt. 12., and it's given by $E_4^3 - E_6^2$.

Up to a constant, this is the same as

$$g_2^3 - 27 g_3^2 = \text{discriminant of } 4x^3 - g_2(\tau)x - g_3(\tau).$$

This modular form is called the <u>modular discriminant</u>, so this modular form is nonvanishing on $\mathcal{H}$.

After normalization so that the leading coeff is 1, we write

$$\Delta(\tau) = q - 24q^2 + 252q^3 + \cdots$$

These coefficients are denoted by $\tau(n)$, i.e. $\Delta(\tau) = \sum_{n\geq 1} \tau(n) q^n$.

<u>Another modular function</u>

<u>Recall</u>: $\mathbb{C}(X(\Gamma)) = A_0(\Gamma)$

<u>Recall</u>: $X(SL_2(\mathbb{Z})) \cong \mathbb{P}^1$

So there must be an isomorphism

$$j: X(SL_2(\mathbb{Z})) \longrightarrow \mathbb{P}^1$$

which we can choose so that $j(\infty) = \infty$.
This is the same as a mero. fn on $X(SL_2(\mathbb{Z}))$ with a simple pole at ∞.

$$j \sim \frac{E_4^3}{\Delta}$$ normalize so that

$$j = 1728 \frac{g_2^3}{g_2^3 - 27g_3^2}.$$

$Y(SL_2(\mathbb{Z}))$ classifies ell. curves up to iso.
Hence, the function

$$j : y^2 = 4x^3 - ax - b \longmapsto 1728 \frac{a^3}{a^3 - 27b^2}$$

is a bijection between iso classes of ell. curves and $\mathbb{C}$.
This is called the j-invariant.

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots$$

$e^{\pi\sqrt{163}}$ is almost an integer

Can show that if $\mathbb{Z}[\alpha]$ is the ring of int's of quadratic number field with class #1, then $j(\alpha) \in \mathbb{Z}$.
Applying this to $\alpha = \frac{1 + \sqrt{-163}}{2}$

$\frac{1}{q}$ gives $e^{\pi\sqrt{163}}$.