Last time

Defined Riemann surface structure on $X(\Gamma) = \Gamma \backslash \mathcal{H}^*$.

Defined...
for $f : \mathcal{H} \longrightarrow \mathbb{P}^1$ meromorphic, $k \in \mathbb{Z}$, $\Gamma \leq SL_2(\mathbb{Z})$ cong. subgp

- f is modular of wt k on Γ if
$$f|_\gamma = f \quad \forall \gamma \in \Gamma$$
where
$$(f|\gamma)(\tau) = \frac{(\det \gamma)^{k/2}}{(c\tau + d)^k} f(\gamma\tau) \qquad \gamma \in GL_n(\mathbb{C}) \text{ if } \gamma\tau \in \mathcal{H}$$
$$= \frac{1}{(c\tau+d)^k} f(\gamma\tau) \qquad \gamma \in SL_2(\mathbb{Z})$$

- q-exp of f at ∞: if h is the width of ∞ then $f(\tau + h) = f(\tau)$, so interpret f as a function on punctured unit disk

$$\mathcal{H} \longrightarrow$$ 
$$\tau \longmapsto e^{2\pi i \tau / h}$$

f corresponds to a holo. fn g on 
$$g(q) = \sum_{n \in \mathbb{Z}} a_n q^n$$
s.t. $f(\tau) = g(e^{2\pi i \tau / h})$.
g is the q-exp of f at ∞.

Let $\operatorname{ord}_\infty(f) = \operatorname{ord}_0(g) = \min\{n : a_n \neq 0\} \in \mathbb{Z} \cup \{\pm\infty\}$.

- f is mero at ∞ if $\operatorname{ord}_\infty(f) > -\infty$
  holo at ∞ $\quad \operatorname{ord}_\infty(f) \geq 0$
  vanishes at ∞ $\quad \operatorname{ord}_\infty(f) > 0$.

- q-exp of f at cusp $s = \gamma\infty$, $\gamma \in SL_2(\mathbb{Z})$, is q-exp of $f|\gamma$ at ∞.
  Similarly define $\operatorname{ord}_s(f) = \operatorname{ord}_\infty(f|\gamma)$.

- $\mathcal{A}_k(\Gamma) = \{$mero mod forms of wt k on $\Gamma\}$ mero at cusps
  $\mathcal{M}_k(\Gamma) = \{$holo ———————$\}$ holo at cusps
  $S_k(\Gamma) = \{$cusp forms ———————$\}$ vanishing at cusps

Defined a map

$$\Omega^{\otimes k}(X(\Gamma)) = \Omega^1(X(\Gamma))^{\otimes k} \longrightarrow \mathcal{A}_{2k}(\Gamma)$$

If $\pi : \mathcal{H} \longrightarrow Y(\Gamma) \hookrightarrow X(\Gamma)$, $\omega \in \Omega^{\otimes k}(X(\Gamma))$
$\pi^*(\omega) = f(\tau)(d\tau)^k$, $f \in \mathcal{A}_{2k}(\Gamma)$

$$\omega \longmapsto f.$$

Today

- Show this map is surjective
- Compare $\mathrm{ord}_\tau(f)$ to $\mathrm{ord}_{\pi(\tau)}(\omega)$, $\tau \in \mathcal{H}^*$
- Compute dimension of $\mathcal{M}_{2k}(\Gamma)$
- Examples of mod forms.

Babby's first modular forms

Broke approach

Want to construct a $|\gamma$-invariant function on $\mathcal{H}$ $\forall \gamma \in \Gamma$.

Obvious approach:

$$f(\tau) = \sum_{\gamma \in SL_2(\mathbb{Z})} (1|\gamma)(\tau) = \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma \in SL_2(\mathbb{Z})} \frac{1}{(c\tau + d)^k}.$$

Problem: if $(c,d) = 1$ there are ∞ly many $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ given by $T^n \begin{pmatrix} a_0 & b_0 \\ c & d \end{pmatrix}$ $n \in \mathbb{Z}$ for some $\begin{pmatrix} a_0 & b_0 \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ (Bezout's lemma).

Try modding out by $\langle T \rangle$:

$$f(\tau) = \sum_{\gamma \in SL_2(\mathbb{Z})/\langle T \rangle} (1|\gamma)(\tau) = \sum_{(c,d)=1} \frac{1}{(c\tau+d)^k}.$$

This works! (If $k>2$).

$$\mathbb{Z}^2 \setminus \{(0,0)\} \longleftrightarrow \mathbb{N} \times \{(c,d) \in \mathbb{Z}^2 : \gcd(c,d) = 1\}$$

$$(nc, nd) \longleftrightarrow (n, (c,d))$$

$$(a,b) \longmapsto \left(\gcd(a,b), \frac{(a,b)}{\gcd(a,b)}\right).$$

so

$$G_k(\tau) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{0,0\}} \frac{1}{(m\tau+n)^k} \quad \text{(absolutely convergent for } k>2\text{)}$$

$$= \sum_{n \in \mathbb{N}} \sum_{\gcd(c,d)=1} \frac{1}{n^k (c\tau+d)^k}$$

$$= \sum_{n \in \mathbb{N}} \frac{1}{n^k} \sum_{\gcd(c,d)=1} \frac{1}{(c\tau+d)^k} = \zeta(k) f(\tau)$$

Woke approach

Modular forms should have something to do with moduli of elliptic curves.

Recall: for $\tau \in \mathcal{H}$, the elliptic curve corresponding to τ

$$\begin{array}{ccc} \mathbb{C}/\Lambda_\tau & & z \\ \cong \big\downarrow \phi & & \big\downarrow \\ & & (\wp_{\Lambda_\tau}(z), \wp'_{\Lambda_\tau}(z)) \\ E_\tau : & y^2 = 4x^3 - g_2(\tau)x - g_3(\tau) & \end{array}$$

Want to relate $g_2(\gamma\tau)$ to $g_2(\tau)$
$g_3(\gamma\tau)$ to $g_3(\tau)$

Recall that $\Lambda_{\gamma\tau} = \frac{1}{c\tau+d} \Lambda_\tau$, so that

$$\mathbb{C}/\Lambda_{\gamma\tau} \xrightarrow{\sim} \mathbb{C}/\Lambda_\tau$$

$$z \longmapsto (c\tau+d)z$$

Let $\lambda = \frac{1}{c\tau+d}$.

$$
\begin{tikzcd}
\mathbb{C}/\lambda\Lambda_\tau \arrow[rrr] \arrow[ddd] & & & \mathbb{C}/\Lambda_\tau \arrow[ddd] \\
 & \lambda z \arrow[r, mapsto] \arrow[d, mapsto] & z \arrow[d, mapsto] & \\
 & (\wp_{\lambda\Lambda_\tau}(\lambda z), \wp'_{\lambda\Lambda_\tau}(\lambda z)) \arrow[r, mapsto] & (\wp_{\Lambda_\tau}(z), \wp'_{\Lambda_\tau}(z)) & \\
E_{\lambda\tau} \arrow[rrr] & & & E_\tau
\end{tikzcd}
$$

What is the blue map?

Let $\{\omega_1, \omega_2\}$ be a basis for Λ_τ

$\{\lambda\omega_1, \lambda\omega_2\}$ basis for $\lambda\Lambda_\tau$

$$\wp_{\Lambda_\tau}(z) = \frac{1}{z^2} + \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(z+m\omega_1+n\omega_2)^2} - \frac{1}{(m\omega_1+n\omega_2)^2}$$

$$\wp_{\lambda\Lambda_\tau}(\lambda z) = \frac{1}{(\lambda z)^2} + \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(\lambda z+m\lambda\omega_1+n\lambda\omega_2)^2} - \frac{1}{(m\lambda\omega_1+n\lambda\omega_2)^2}$$

$$\wp_{\lambda\Lambda_\tau}(\lambda z) = \frac{1}{\lambda^2}\wp_{\Lambda_\tau}(z)$$

also $\wp'_{\lambda\Lambda_\tau}(\lambda z) = \frac{1}{\lambda^3}\wp_{\Lambda_\tau}(z)$.

The blue map is

$$\left(\wp_{\lambda\Lambda_\tau}(\lambda z),\ \wp'_{\lambda\Lambda_\tau}(\lambda z)\right) \longmapsto \left(\wp_{\Lambda_\tau}(z),\ \wp'_{\Lambda_\tau}(z)\right)$$

$$\parallel$$

$$\left(\frac{1}{\lambda^2}\wp_{\Lambda_\tau}(z),\ \frac{1}{\lambda^3}\wp'_{\Lambda_\tau}(z)\right)$$

$$(x,y) \longmapsto (\lambda^2 x,\ \lambda^3 y)$$

In other words, there is an iso

$$\begin{array}{ccl} E_{\gamma\tau} & : & y^2 = 4x^3 - g_2(\gamma\tau)x - g_3(\gamma\tau) \\ \Big\downarrow & & \begin{array}{c}(x,y) \\ \downarrow \\ (\lambda^2 x, \lambda^3 y)\end{array} \\ E_\tau & : & y^2 = 4x^3 - g_2(\tau)x - g_3(\tau) \end{array}$$

So the solutions of

$$\frac{1}{\lambda^6}y^2 = 4\frac{1}{\lambda^6}x^3 - \frac{1}{\lambda^2}g_2(\gamma\tau)x - g_3(\gamma\tau) \quad \text{and}$$

$$y^2 = 4x^3 - g_2(\tau)x - g_3(\tau)$$

are the same.

Hence

$$g_2(\gamma\tau) = \lambda^{-4} g_2(\tau) = (c\tau+d)^4 g_2(\tau)$$

$$g_3(\gamma\tau) = \lambda^{-6} g_3(\tau) = (c\tau+d)^6 g_3(\tau)$$

so g_2 is modular of wt. 4
g_3 is modular of wt. 6.

Both the broke and woke approach give (up to a constant) the <u>Eisenstein series</u>.

<u>Defn</u> The Eisenstein series of wt. $k>2$ is

$$G_k(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m\tau+n)^k}.$$

This is a wt. k modular form on $SL_2(\mathbb{Z})$.
It is 0 when k is odd.

<u>Modular forms and differentials</u>

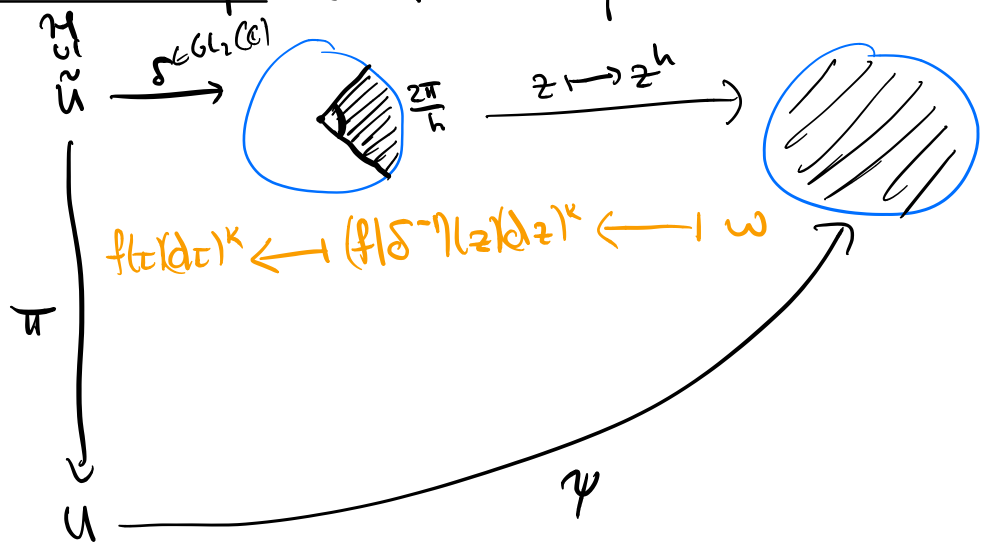
$$\Omega^{\otimes k}(X(\Gamma)) \longrightarrow A_{2k}(\Gamma)$$

$$\omega = \text{``}f(\tau)(d\tau)^k\text{''} \longmapsto f$$

Constructed this last time. Now we show that this is surjective.

Let $f \in A_{2k}(\Gamma)$. We will construct $\omega \in \Omega^{\otimes k}(X(\Gamma))$ with $\pi^*(\omega) = f(\tau)(d\tau)^k$ locally.

<u>At non-cusps</u> $\tau \in \mathcal{H}$ with period $h = h_\tau$.



$\cap$

$Y(\Gamma)$

Need $\omega \in \Omega^{\otimes k}(\text{disk})$ which pulls back to $(f|\delta^{-1})(z)(dz)^k$ under $z \mapsto z^h$.

<u>Recall:</u> $\delta \Gamma \delta^{-1} = \langle z \longmapsto \zeta_h z \rangle$.

Since $(f|\delta^{-1})(z)(dz)^k$ is $\delta\Gamma\delta^{-1}$-invariant, we have

$$(f|\delta^{-1})(\zeta_h z)\,(d(\zeta_h z))^k = (f|\delta^{-1})(z)(dz)^k$$

$$\zeta_h^k\,(f|\delta^{-1})(\zeta_h z)(dz)^k = (f|\delta^{-1})(z)(dz)^k$$

$$\Longrightarrow \quad \zeta_h^k\,(f|\delta^{-1})(\zeta_h z) = (f|\delta^{-1})(z)$$

$$\Longrightarrow \quad (\zeta_h z)^k\,(f|\delta^{-1})(\zeta_h z) = z^k\,(f|\delta^{-1})(z)$$

$$\Longrightarrow \quad z^k(f|\delta^{-1})(z) = g(z^h)$$

for some holo. g on .

Let $\omega = \dfrac{g(q)}{(hq)^k}(dq)^k$.

Then

$$\omega \xmapsto{(z\mapsto z^h)^*} \frac{g(z^h)}{(hz^h)^k}(dz^h)^k$$

$$= \frac{z^k(f|\delta^{-1})(z)}{h^k z^{hk}}\, h^k z^{k(h-1)}(dz)^k$$

$$= (f|\delta^{-1})(z)(dz)^k$$

$$\xmapsto{\delta^*} f(\tau)(d\tau)^k.$$

<u>Note:</u> $\operatorname{ord}_\tau f = \operatorname{ord}_0(f|\delta^{-1}) = \operatorname{ord}_0\left(\dfrac{g(z^h)}{z^k}\right)$

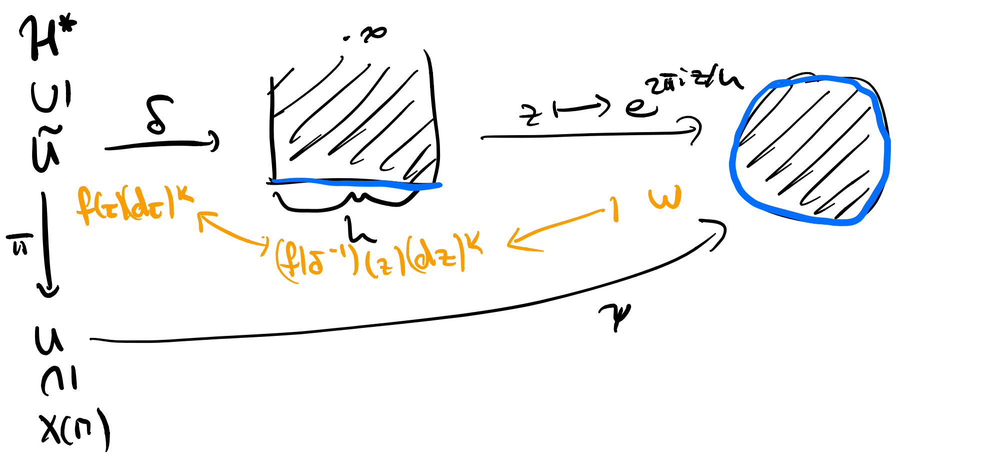
$$= h \operatorname{ord}_0 g - k$$

$$\operatorname{ord}_0 \omega = \operatorname{ord}_0\left(\frac{g(q)}{(hq)^k}\right) = \operatorname{ord}_0 g - k$$

So

$$\operatorname{ord}_\tau f = h \operatorname{ord}_0 \omega + k(h-1).$$

<u>At cusp</u> s, let h be the width of s.



Want to construct ω which pulls back to $(f|\delta^{-1})(z)(dz)^k$.

Let $g(q)$ be the q-exp of f at s
$= $ the q-exp of $f|\delta^{-1}$ at ∞

By defn $(f|\delta^{-1})(z) = g(e^{2\pi i z/h})$.

Let $\quad \omega = \dfrac{g(q)}{\left(\frac{2\pi i}{h} q\right)^k} (dq)^k$.

$$dq \longmapsto \frac{2\pi i}{h} e^{2\pi i z/h} dz$$

Then

$$\omega \longmapsto \frac{g(e^{2\pi i z/h})}{\left(\frac{2\pi i}{h} e^{2\pi i z/h}\right)^k} \left(\frac{2\pi i}{h}\right)^k (e^{2\pi i z/h})^k (dz)^k$$

$$= (f|\delta^{-1})(z)(dz)^k \xmapsto{\delta^*} f(\tau)(d\tau)^k .$$

Note

$\mathrm{ord}_s f = \mathrm{ord}_0\, g$ by defn

$\mathrm{ord}_0\, \omega = \mathrm{ord}_0 \left(\dfrac{g(q)}{\left(\frac{2\pi i}{h} q\right)^k} \right) = \mathrm{ord}_0\, g - k$

so $\quad \mathrm{ord}_s f = \mathrm{ord}_0\, \omega + k$.

<u>Prop</u> There is an iso of $\mathbb{C}$-vector spaces

$$A_{2k}(\Gamma) \longrightarrow \Omega^{\otimes k}(X(\Gamma))$$
$$f \longmapsto \text{“}f(\tau)(d\tau)^k\text{”}$$

s.t.

$\mathrm{ord}_\tau f = h_\tau \mathrm{ord}_{\pi(\tau)} \omega + k(h_\tau - 1) \qquad$ if $\tau \in \mathcal{H}$

$\mathrm{ord}_s f = \mathrm{ord}_{\pi(s)} \omega + k \qquad$ if $s \in \mathbb{P}^1(\mathbb{Q})$.