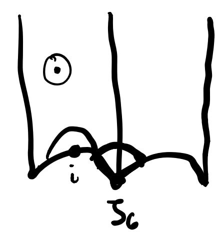
Last time

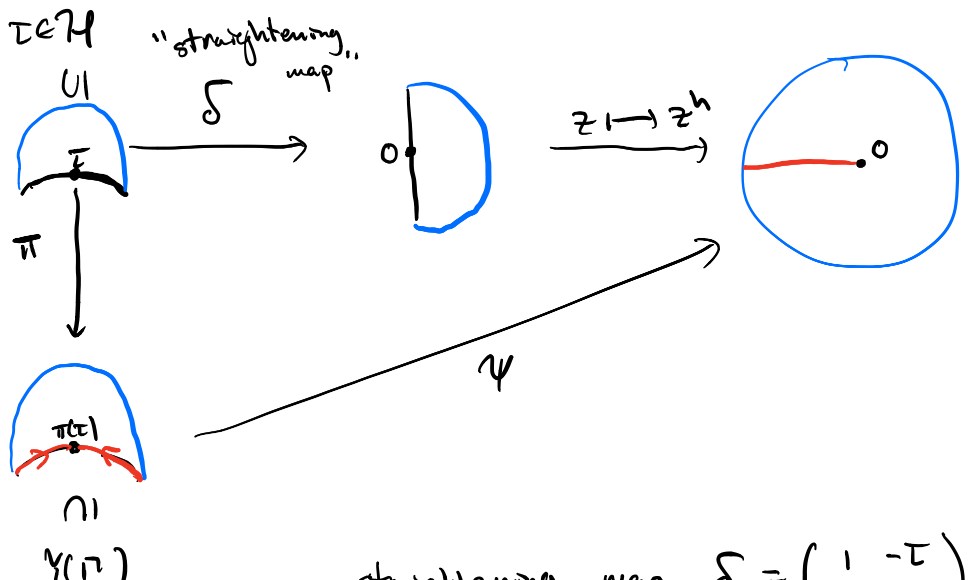
Riemann surface structure on

$$Y(\Gamma) = \Gamma \backslash \mathcal{H} \qquad \Gamma \leq SL_2(\mathbb{Z}) \text{ congruence subgp}$$
$$\cap \qquad\quad \cap \qquad\qquad \mathcal{H} : \text{upper half plane}$$
$$X(\Gamma) = \Gamma \backslash \mathcal{H}^* \qquad \mathcal{H}^* = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$$

$X(\Gamma)$ compact.



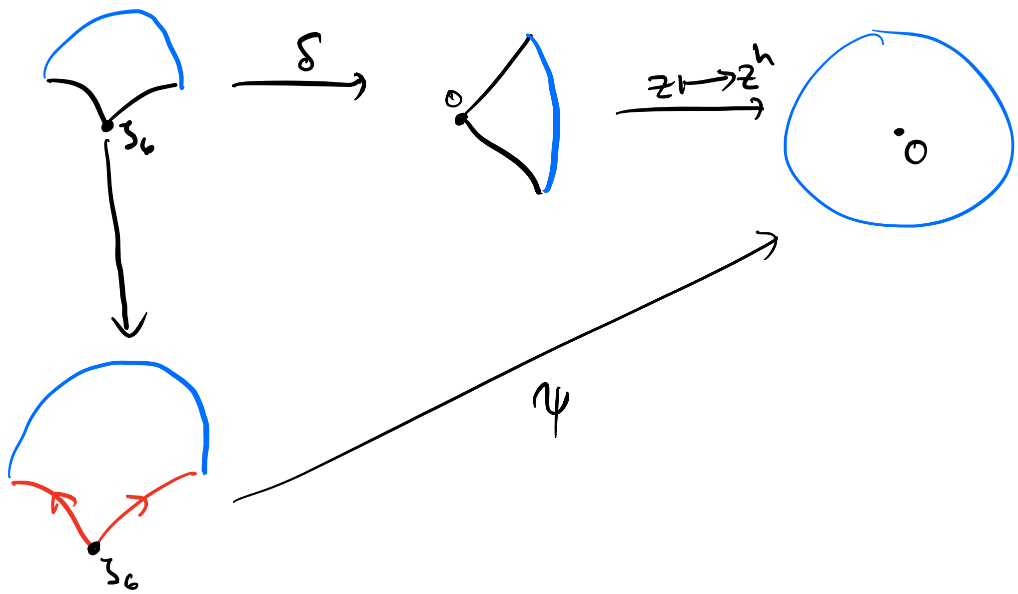
Gave coord. nbhds of non-cusp pts.



straightening map $\delta = \begin{pmatrix} 1 & -\tau \\ 1 & -\bar{\tau} \end{pmatrix} \in GL_2(\mathbb{C})$

$$\delta z = \frac{z - \tau}{z - \bar{\tau}}$$

$$\delta \tau = 0$$
$$\delta \bar{\tau} = \infty$$



$\cap$

$Y(\Gamma)$

Charts at the cusps

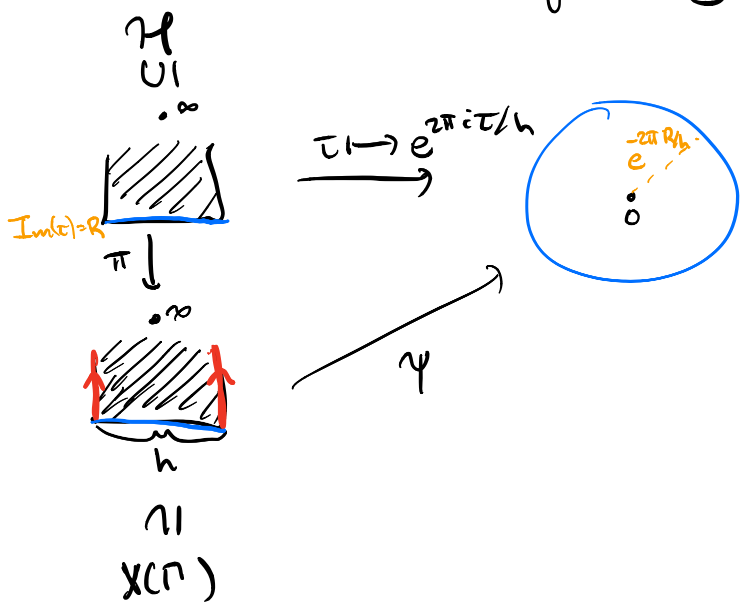
Let $h \in \mathbb{N}$ be minimal s.t. $T^h \in \Gamma$

(Recall $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.)

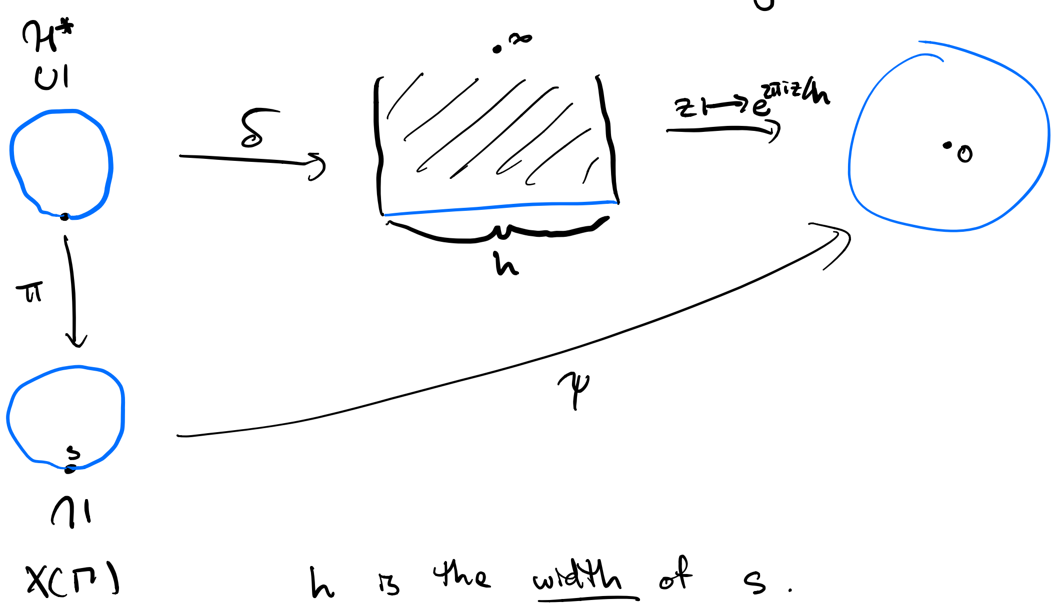
$T^k \in \Gamma$ for some k because $\underset{T^N}{\underset{\parallel}{\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}}} \in \Gamma(N) \subseteq \Gamma$.

h is the <u>width</u> of ∞.

A chart at ∞ is given by



At a general cusp $s \in X(\Gamma) \setminus Y(\Gamma)$, let $\delta \in SL_2(\mathbb{Z})$ be s.t. $\delta s = \infty$. The chart is given by



h is the width of s.

i.e. h is minimal s.t.

$$T^h \in \delta \Gamma \delta^{-1}$$

Exercise: Check that transition maps are holo.

This turns $X(\Gamma)$ into a R.S.!

Rmk All of these charts have the form

$$\begin{tikzcd}
\mathcal{H}^* & & \\
\tilde{U} \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, "\delta"] \arrow[d, "\pi"'] & \delta\tilde{U} \arrow[r, "\rho", "\text{powering or exp}"'] & \odot \\
U \arrow[urr, "\psi"'] & & \\
X(\Gamma) \arrow[u, phantom, sloped, "\subseteq"] & &
\end{tikzcd}$$

Convention: τ is coord on $\tilde{U}$

z is coord on $\delta\tilde{U}$

q is coord on $\odot$.

Modular forms

(Modular curves + differential forms)

Motivation:

Let $\omega \in \Omega^1(Y(\Gamma))$. Locally,

$\omega = f(\tau)d\tau \qquad f \in \mathbb{C}(Y(\Gamma))$

mero. fns on $Y(\Gamma)$

Defn If $f: X \longrightarrow Y$ map of R.S.s. $\omega \in \Omega^1(Y)$, then $f^*\omega \in \Omega^1(X)$ is defined by

$$f^*(dg) = d(g \circ f) \qquad \text{for } g \in \mathcal{O}(Y).$$

Since $\omega \in \Omega^1(Y(\Gamma))$, it is invariant under pullback by $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

$$f(\tau)d\tau = \gamma^*(f(\tau)\,d\tau) = f(\gamma\tau)\,d(\gamma\tau)$$

$$= f(\gamma\tau)\,d\left(\frac{a\tau+b}{c\tau+d}\right)$$

$$= f(\gamma\tau)\left(\frac{(c\tau+d)a - (a\tau+b)c}{(c\tau+d)^2}\right)d\tau$$

$$= f(\gamma\tau)\frac{1}{(c\tau+d)^2}\,d\tau$$

$\Longrightarrow f(\tau) = f(\gamma\tau)\dfrac{1}{(c\tau+d)^2} \Longrightarrow f$ is modular of wt 2.

Defn If $k \in \mathbb{Z}$, $\gamma \in GL_2(\mathbb{C})$, $f: \mathcal{H} \longrightarrow \mathbb{P}^1$ is a function,

let $$(f|\gamma)(\tau) = (c\tau+d)^{-k} f(\gamma\tau).$$

Usually $\gamma \in SL_2(\mathbb{Z})$.

Defn Let $f : \mathcal{H} \longrightarrow \mathbb{P}^1$ be a mero. fn, $\Gamma \leq SL_2(\mathbb{Z})$ be a congruence subgp, $k \in \mathbb{Z}$.

(1) f is modular of weight k on Γ if

$$f|_\gamma = f \quad \forall \gamma \in \Gamma.$$

$$\left[f(\gamma\tau) = (c\tau+d)^k f(\tau) \quad \forall \tau \in \mathcal{H},\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. \right]$$

(2) The q-expansion of f at ∞ is defined as follows. Let h be the width of the cusp $\infty \in X(\Gamma)$. Then

$f(\tau+h) = (f|T^h)(\tau) = f(\tau)$, so f factors through the map

$$\mathcal{H} \xrightarrow[\tau \mapsto e^{2\pi i \tau/h}]{} D = \{ z : |z| < 1 \}.$$

$$e^{2\pi i\tau/h} = q_h$$

Thus $f(q)$ is meromorphic on $D \setminus \{0\}$, so we can write

$$f(q) = \sum_{n \in \mathbb{Z}} a_n q_h^n \qquad a_n \in \mathbb{C}.$$

This is the q-exp at ∞.

f is meromorphic at ∞ if $a_n = 0$ for $n \ll 0$.

holomorphic at ∞ if $a_n = 0$ for $n < 0$.

vanishes at ∞ if $a_n = 0$ for $n \leq 0$.

(3) The q-expansion of f at the cusp $s \in X(\Gamma) \setminus Y(\Gamma)$ is the q-expansion of $f|\gamma$ where $\gamma \in SL_2(\mathbb{Z})$ is s.t. $\gamma\infty = s$.

f is meromorphic
holomorphic
vanishing
at s if $f|\gamma$ is so at ∞.

(4) f is a <u>meromorphic modular form</u> of wt k on Γ if f is meromorphic at every cusp.

f is a (holomorphic) <u>modular form</u> if f is holomorphic on $\mathcal{H}$ and holo. at every cusp.

f is a <u>cusp form</u> if f is holo on $\mathcal{H}$ and vanishes at every cusp.

$$\underset{\text{mero. m.f.}}{\mathcal{A}_k(\Gamma)} \supseteq \underset{\text{mod. forms.}}{\mathcal{M}_k(\Gamma)} \supseteq \underset{\text{cusp forms}}{S_k(\Gamma)}$$

<u>Note:</u> $\mathcal{A}_0(\Gamma) = \mathbb{C}(X(\Gamma))$.

Goal: see how $f \in \mathcal{A}_k(\Gamma)$ are "the same as" diff'ls on $X(\Gamma)$ for $k \in 2\mathbb{Z}$.

Let X be a R.S.

<u>Defn</u> $\Omega^k(X) = \Omega^1(X)^{\otimes k}$. So elts of $\Omega^k(X)$ locally look like

$$f(q)(dq)^k \quad \text{where } f \in \mathbb{C}(X) \text{ and } q \text{ is a}$$

local coord.

(Note: not the exterior power!)

Our main theorem is an isomorphism of $\mathbb{C}$-spaces

$$\begin{aligned} A_{2k}(\Gamma) &\longrightarrow \Omega^k(X(\Gamma)) \\ \text{"} f &\longmapsto f(\tau)(d\tau)^k \text{"} \end{aligned}$$

Doesn't formally make sense: $f \notin \mathbb{C}(X(\Gamma))$.

Need to work locally.

Let $\pi: \mathcal{H} \longrightarrow Y(\Gamma) \hookrightarrow X(\Gamma)$

$$\begin{aligned} \pi^*: \Omega^k(X(\Gamma)) &\longrightarrow \Omega^k(\mathcal{H}) \\ \omega &\longmapsto f(\tau)(d\tau)^k \quad \text{some } f \in \mathbb{C}(\mathcal{H}). \end{aligned}$$

By the commutativity of

$$\begin{tikzcd} \mathcal{H} \arrow[r, "\pi"] \arrow[d, "\gamma"'] & X(\Gamma) \arrow[d, "id"] \\ \mathcal{H} \arrow[r, "\pi"'] & X(\Gamma) \end{tikzcd}
\qquad
\begin{tikzcd} f(\tau)d\tau^k & \omega \arrow[l, mapsto] \\ f(\tau)d\tau^k \arrow[u, mapsto] & \omega \arrow[l, mapsto] \arrow[u, mapsto] \end{tikzcd}$$

we have

$$f(\gamma\tau)(d(\gamma\tau))^k = f(\tau)\,d\tau$$

$$\| $$

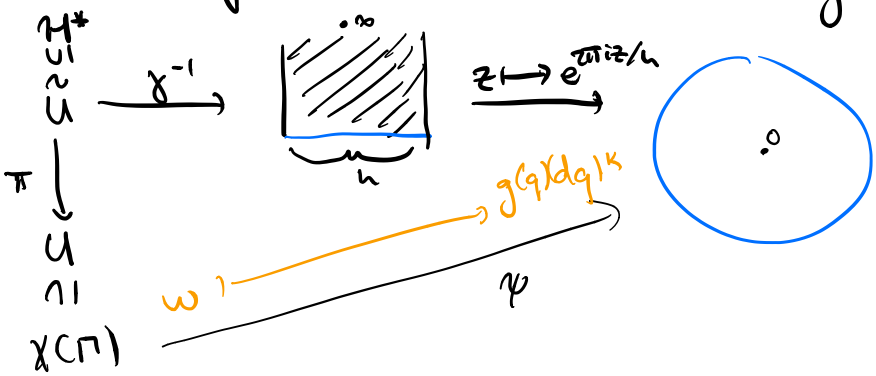
$$f(\gamma\tau)\left(\frac{1}{c\tau+d}\right)^{2k}(d\tau)^k$$

$\Longrightarrow$ $(f|\gamma)(\tau) = f(\tau)$ ($|_\gamma$ is of wt. $2k$).

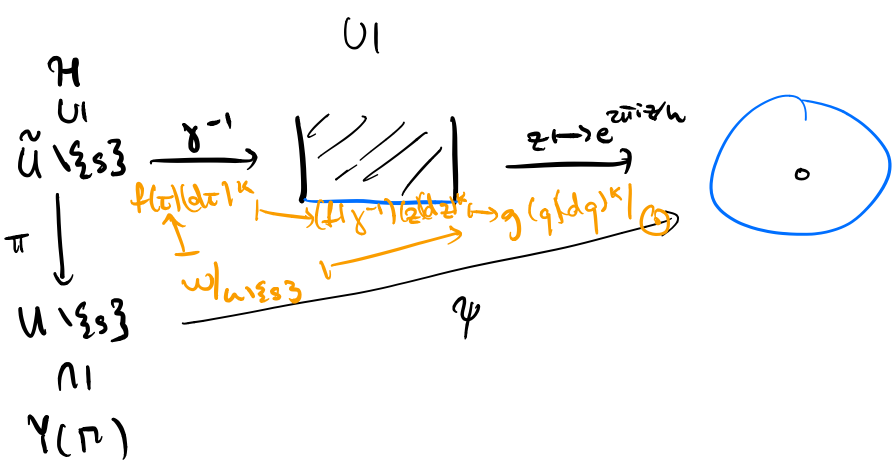
To show $f \in A_{2k}(\Gamma)$, need to check it is mero at the cusps.

Suffices to show $f|_\gamma$ is mero at ∞ $\forall \gamma \in SL_2(\mathbb{Z})$.

Let $s = \gamma\infty$. The chart at s is given by



g is mero. at 0!



we've shown $(f|\gamma^{-1})(z)(dz)^k = g(q)(dq)^k$.
Want to relate $(f|\gamma^{-1})(z)$ to $g(q)$.

$$q = e^{2\pi i z/h}$$
$$dq = \frac{2\pi i}{h} e^{2\pi i z/h} dz$$
$$dq = \frac{2\pi i}{h} q \, dz$$
$$(f|\gamma^{-1})(z)(dz)^k = g(q)\left(\frac{2\pi i}{h} q\right)^k (dz)^k$$

so $(f|\gamma^{-1})(z) = \left(\frac{2\pi i}{h}\right)^k g(q) q^k$

is mero at ∞ since $g(q)$ is mero. at 0.

So we have a map

$$\begin{aligned} \Omega^k(X(\Gamma)) &\longrightarrow A_{2k}(\Gamma) \\ \omega = \text{"}f(\tau)(d\tau)^k\text{"} &\longmapsto f(\tau) \end{aligned}$$

Next time: show this is surj.
Examples of modular forms.