Last time

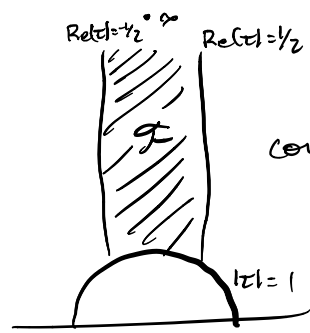
$\Gamma \leq SL_2(\mathbb{Z})$ congruence subgroup

(i.e. $\Gamma \geq \Gamma(N)$ some $N \geq 1$).

$Y(\Gamma) = \Gamma \backslash \mathcal{H}$ $\qquad$ $\mathcal{H}^* = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q}) = \mathcal{H} \cup \mathbb{Q} \cup \{\infty\}$

$\cap \qquad \cap$

$X(\Gamma) = \Gamma \backslash \mathcal{H}^*$



compact

Fundamental domain for $X(SL_2(\mathbb{Z}))$:

Fundamental domain for $X(\Gamma) = \bigcup_{\gamma_i} \gamma_i \mathfrak{F}$

$\{\gamma_i\}$ coset reps for $\underbrace{\Gamma \backslash SL_2(\mathbb{Z})}$.

$[SL_2\mathbb{Z} : \Gamma] \leq [SL_2\mathbb{Z} : \Gamma(N)] < \infty$

$SL_2(\mathbb{Z})$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$.

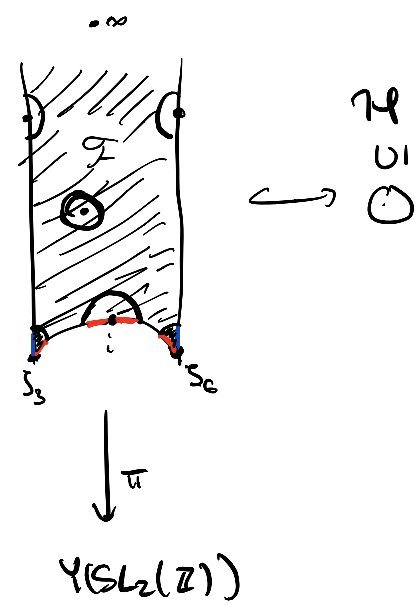
$X(\Gamma) \setminus Y(\Gamma)$ is finite (Γ-orbits of $\mathbb{P}^1(\mathbb{Q})$).

These pts are called <u>cusps</u>.

<u>Prop</u> $X(\Gamma)$ is a compact connected Hausdorff space.
$Y(\Gamma) \subseteq X(\Gamma)$ is an open subspace.

<u>Today</u> Riemann surface structure on $X(\Gamma)$.
Need a coord nbhd at each pt.

Usually easy:



Issue: at most pts $\tau \in \mathcal{H}$, the only $\gamma \in \Gamma$ fixing τ are $\pm I$ (both of which act as the identity on $\mathcal{H}$).
But some $\tau \in \mathcal{H}$ are fixed by more matrices.

<u>Lemma</u>: For $\tau \in \mathcal{H}$, we have

$$\mathrm{Stab}_\tau(SL_2(\mathbb{Z})) \cong \begin{cases} \mathbb{Z}/4\mathbb{Z} & \text{if } \tau \equiv i \pmod{SL_2\mathbb{Z}} \\ \mathbb{Z}/6\mathbb{Z} & \text{if } \tau \equiv \zeta_3 \pmod{SL_2\mathbb{Z}} \\ \mathbb{Z}/2\mathbb{Z} & \text{else} \end{cases}.$$

<u>Pf.</u> Suppose that $\tau \in \mathcal{F}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \tau$

some $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

$$\mathrm{Im}(\tau) = \mathrm{Im}(\gamma\tau) = \frac{\mathrm{Im}(\tau)}{|c\tau+d|^2}, \quad \text{so} \quad |c\tau + d| = 1.$$

$c, d \in \mathbb{Z}$, and since $\tau \in \mathcal{F}$ we have $|\tau| \geq 1$.

So either $c = \pm 1$ and $|\tau| = 1$

or

$d = \pm 1$ and $c = 0$.

If $d = 1$, $c = 0$: $a = 1$, $\tau = \tau + b$

$$\Rightarrow b = 0$$

$$\Rightarrow \gamma = \pm I.$$

If $c = \pm 1$, $|\tau| = 1$: either $d = 0$ or $d = \pm 1$

- if $d = 0$: $c = 1$, $b = -1$

$$\frac{a\tau - 1}{\tau} = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \tau = \tau$$

$$\Rightarrow \tau^2 - a\tau + 1 = 0$$

$$\Rightarrow \tau = \frac{a + \sqrt{a^2 - 4}}{2}$$

$$\Rightarrow a = 0 \text{ or } \pm 1$$

$\tau = i$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$\tau = \zeta_6, \zeta_3$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm \begin{pmatrix} \pm 1 & -1 \\ 1 & 0 \end{pmatrix}$$

- if $d = \pm 1$: if $|\tau| = 1$ and $|\pm\tau \pm 1| = 1$
then $\tau = \zeta_6, \zeta_3$

[arithmetic...]

$$\mathrm{Stab}_\tau(SL_2\mathbb{Z}) = \begin{cases} \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle & \tau \equiv i \\ \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \right\rangle & \tau \equiv \zeta_3 \\ \langle -I \rangle & \text{else} \end{cases}$$

□

Cor If $\Gamma \leq SL_2(\mathbb{Z})$ is a congruence subgp then $\mathrm{Stab}_\tau \Gamma$ is finite cyclic.

Pf. $\mathrm{Stab}_\tau \Gamma \leq \mathrm{Stab}_\tau(SL_2(\mathbb{Z}))$. □

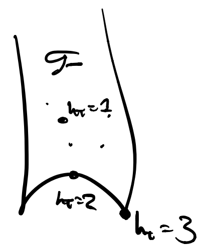
$-I = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ acts as the identity

$$SL_2(\mathbb{Z}) \twoheadrightarrow SL_2(\mathbb{Z})/\{\pm I\} \curvearrowright \mathcal{H}$$
$$\| $$
$$PSL_2(\mathbb{Z})$$

Defn Let $\Gamma \leq SL_2(\mathbb{Z})$ be a congruence subgp. For $\tau \in \mathcal{H}$ defined the *period* of τ to be

$$h_\tau = \#\left(\{\pm I\}\mathrm{Stab}_\tau \Gamma / \{\pm I\}\right) = \begin{cases} \#\mathrm{Stab}_\tau \Gamma / 2 & \text{if } -I \in \Gamma \\ \#\mathrm{Stab}_\tau \Gamma & \text{if } -I \notin \Gamma. \end{cases}$$

Since $\mathrm{Stab}_{\gamma\tau}\Gamma = \mathrm{Stab}_\tau(\gamma\Gamma\gamma^{-1})$, h_τ only depends on the image of τ in $Y(\Gamma)$.

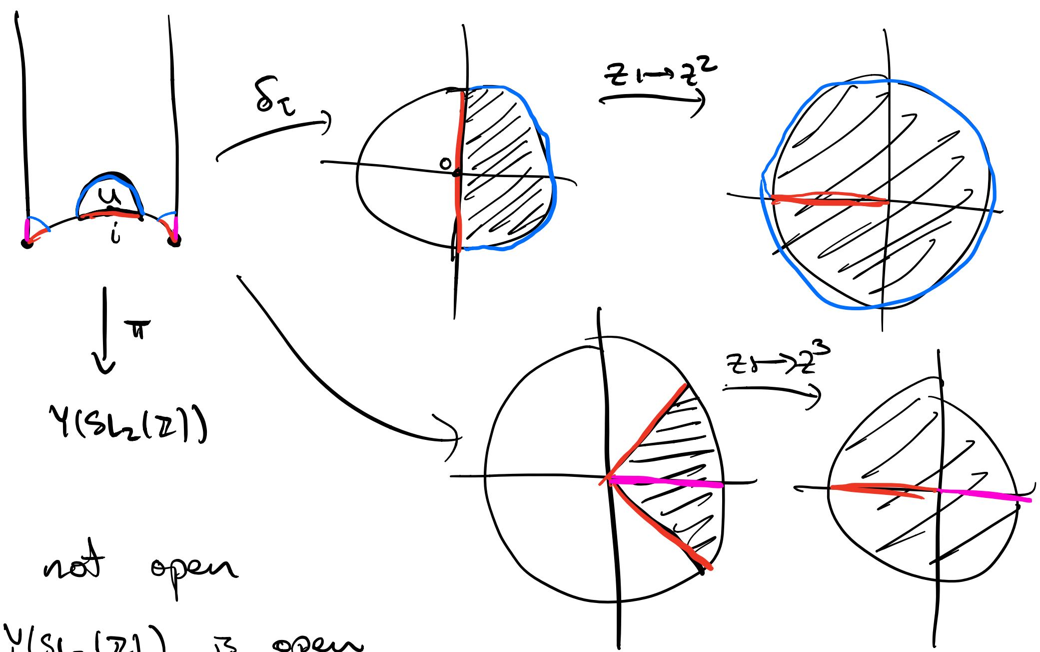


Call τ <u>elliptic</u> if $h_\tau > 1$.

<u>Note</u>: $Y(\Gamma)$ has only finitely many elliptic points.

Coord. nbhds at elliptic pts

<u>Idea</u>:



$U \subseteq \mathcal{H}$ not open

$\pi(U) \subseteq Y(SL_2(\mathbb{Z}))$ is open

Consider

$$\mathbb{P}^1 \xrightarrow{\delta_\tau} \mathbb{P}^1$$
$$z \longmapsto \frac{z-\tau}{z-\bar{\tau}}$$
$$\tau \longmapsto 0$$
$$\bar{\tau} \longmapsto \infty$$

$$\delta_\tau = \begin{pmatrix} 1 & -\tau \\ 1 & -\bar{\tau} \end{pmatrix} \in GL_2(\mathbb{C})$$

$$\mathrm{Stab}_0\left(\delta_\tau\{\pm I\}\Gamma\delta_\tau^{-1}/\{\pm I\}\right) = \delta_\tau\,\mathrm{Stab}_\tau\left(\{\pm I\}\Gamma/\{\pm I\}\right)\delta_\tau^{-1}$$

is cyclic of order h_τ, and consists of rational maps $z \longmapsto \frac{az+b}{cz+d}$ fixing $0, \infty$. Any such rational map is $z \longmapsto \lambda z$, $\lambda \in \mathbb{C}^\times$. Hence this stabilizer is

$$\left\{ z \longmapsto \lambda z : \lambda \in \frac{2\pi i}{h_\tau}\mathbb{Z} \right\}.$$

Suppose $\tau \in \mathcal{H}$ with $h_\tau > 1$. Recall the lemma:

<u>Lemma</u> If $\tau, \tau' \in \mathcal{H}$, $\exists$ nbhds U, U' of τ, τ' s.t. if $\gamma \in \Gamma$ with $\gamma U \cap U' \neq \emptyset$ then $\gamma\tau = \tau'$.

Applying the lemma with $\tau = \tau'$,

<u>Corollary</u> If $\tau \in \mathcal{H}$, then $\exists$ nbhd $U \ni \tau$ s.t. if $\gamma \in \Gamma$ with $\gamma U \cap U \neq \emptyset$ then $\gamma \in \mathrm{Stab}_\tau \Gamma$.

Prop The following diagram gives a homeomorphism φ. Let $U \subseteq \mathcal{H}$ be as in corollary.

$$\begin{tikzcd}
U \subseteq \mathbb{C} \arrow[r, "\delta_\tau"] \arrow[d, "\pi"'] & \delta_\tau(U) \subseteq \mathbb{C} \arrow[r, "z \mapsto z^{h_\tau}"] & V \subseteq \mathbb{C} \\
\pi(U) \subseteq Y(\Gamma) \arrow[urr, "\varphi"'] & &
\end{tikzcd}$$

Pf. Well-defined + injective: let $\tau_1, \tau_2 \in U$

$$\begin{aligned}
\pi(\tau_1) = \pi(\tau_2) &\iff \tau_1 = \gamma\tau_2 \quad \text{some } \gamma \in \mathrm{Stab}_\tau \Gamma \\
&\iff \delta_\tau\tau_1 = (\delta_\tau \gamma \delta_\tau^{-1})(\delta_\tau\tau_2) \quad \gamma \in \mathrm{Stab}_\tau \Gamma \\
&\iff \delta_\tau\tau_1 = \zeta_{h_\tau}^{d}\, \delta_\tau\tau_2 \quad d \in \mathbb{Z} \\
&\iff (\delta_\tau\tau_1)^{h_\tau} = (\delta_\tau\tau_2)^{h_\tau}
\end{aligned}$$

Surj: by definition

homeomorphism: π is an open map, so is δ_τ, $z \mapsto z^{h_\tau}$.

$\square$

<u>Exercise</u> Check that the transition maps are holomorphic.