Last time

Modular curves:

subgroups $\quad \Gamma(N) \leq \Gamma_1(N) \leq \Gamma_0(N) \leq SL_2(\mathbb{Z})$

$\quad\quad\quad\quad\quad S(N) \quad S_1(N) \quad S_0(N)$

$$\Gamma \backslash \mathcal{H} \xrightarrow{\sim} S$$

$\uparrow$ set of elliptic curves with extra structure.

Eg. $S_1(N) = \left\{ (E,Q) : \begin{array}{l} E \text{ ell. curve,} \\ Q \in E[N] \text{ of order } N \end{array} \right\}$.

"$\Gamma \backslash \mathcal{H}$ is a moduli space of elliptic curves"

Today: more about $\Gamma \backslash \mathcal{H}$.

Why? Suppose ω is a differential on $SL_2(\mathbb{Z}) \backslash \mathcal{H}$.

$\uparrow$ z

$$\omega = f(z)\,dz$$

$$\gamma^* \omega = f(\gamma z)\, d(\gamma z) = f(z)\,dz \qquad \gamma \in SL_2(\mathbb{Z})$$

$$= f(\gamma z) \frac{1}{(cz+d)^2}\, dz$$

$$f(\gamma z) = (cz+d)^2 f(z) \qquad \text{wt 2 modular form}$$

Defn A <u>congruence subgroup</u> of $SL_2(\mathbb{Z})$ is a subgp Γ containing $\Gamma(N) = \{ \gamma \in SL_2(\mathbb{Z}) : \gamma \equiv I \pmod N \}$ for some $N \geq 1$.

We will study $\Gamma\backslash\mathcal{H}$ for congruence subgps Γ.

Topology on $\Gamma\backslash\mathcal{H}$ is defined via the map

$\pi: \mathcal{H} \longrightarrow \Gamma\backslash\mathcal{H}$.

That is, we define $U \subseteq \Gamma\backslash\mathcal{H}$ to be open iff $\pi^{-1}(U)$ is open.

To study $\Gamma\backslash\mathcal{H}$, let's look at another perspective on $\mathcal{H}$.

$$\begin{aligned} SL_2(\mathbb{R}) &\longrightarrow \mathcal{H} \\ \gamma &\longmapsto \gamma i. \end{aligned}$$

This is continuous, and

$$\begin{aligned} \mathrm{Stab}_i(SL_2(\mathbb{R})) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) : \frac{ai+b}{ci+d} = i \right\} \\ &= \left\{ \underline{\qquad\qquad} : ai+b = -c+di \right\} \\ &= \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a^2+b^2=1 \right\} \\ &= \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\} = SO_2(\mathbb{R}). \end{aligned}$$

Thus we have a continuous injection

$$SL_2(\mathbb{R})/SO_2(\mathbb{R}) \longrightarrow \mathcal{H}$$

with continuous inverse

$$s : \mathcal{H} \longrightarrow SL_2(\mathbb{R})/SO_2(\mathbb{R})$$
$$x+iy \longmapsto \frac{1}{\sqrt{y}}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \pmod{SO_2(\mathbb{R})}.$$

Hence

$$s : \mathcal{H} \xrightarrow{\sim} SL_2(\mathbb{R})/SO_2(\mathbb{R})$$

is a $SL_2(\mathbb{R})$-equivariant homeomorphism.

<u>Prop</u> $\Gamma\backslash\mathcal{H}$ is Hausdorff.

<u>Pf.</u> Let $\tau, \tau' \in \mathcal{H}$ with $\pi(\tau), \pi(\tau') \in \Gamma\backslash\mathcal{H}$ distinct. We want disjoint nbhds $\pi(U), \pi(U')$ of $\pi(\tau), \pi(\tau')$. This is equivalent to finding nbhds U, U' of τ, τ' s.t. $\Gamma U \cap U' = \emptyset$.

Lemma Any $\tau, \tau' \in \mathcal{H}$ have nbhds U, U' s.t. if $\gamma \in \Gamma$ with $\gamma U \cap U' \neq \emptyset$ then $\gamma\tau = \tau'$.
(Note $\tau = \tau'$ is allowed.)

Pf. If true for $SL_2(\mathbb{Z})$, true for $\Gamma \leq SL_2(\mathbb{Z})$. So take $\Gamma = SL_2(\mathbb{Z})$.

Let U_0, U_0' of τ, τ' with compact closure. We will show that
$$\gamma U_0 \cap U_0' \neq \emptyset$$
for only finitely many $\gamma \in SL_2(\mathbb{Z})$, and then shrink U_0, U_0' to finish.

Using $s : \mathcal{H} \xrightarrow{\sim} SL_2(\mathbb{R})/SO_2(\mathbb{R})$ from above we have for $z, z' \in \mathcal{H}$, $\gamma \in SL_2(\mathbb{Z})$

$$\gamma z = z' \iff \gamma s(z) \equiv s(z') \pmod{SO_2(\mathbb{R})}$$
$$\iff \gamma \in s(z') SO_2(\mathbb{R}) s(z)^{-1}.$$

Thus,
$$\gamma U_0 \cap U_0' \neq 0 \implies \gamma \overline{U_0} \cap \overline{U_0'} \neq 0$$
$$\implies \gamma \in \underbrace{\underbrace{s(\overline{U_0'}) SO_2(\mathbb{R}) s(\overline{U_0})^{-1}}_{\text{compact}} \cap \underbrace{SL_2(\mathbb{Z})}_{\text{discrete}}}_{\text{finite.}}$$

So

$$S = \{\gamma \in SL_2(\mathbb{Z}) : \gamma U_0 \cap U_0' \neq \emptyset, \gamma\tau \neq \tau'\}$$

is finite.

Since $\mathcal{H}$ is Hausdorff, so we may pick for each $\gamma \in S$ disjoint nbhds

$$U_{1,\gamma} \ni \gamma(\tau), \quad U_{1,\gamma}' \ni \tau'.$$

Let

$$U = U_0 \cap \left(\bigcap_{\gamma \in S} \gamma^{-1} U_{1,\gamma} \right)$$

$$U' = U_0' \cap \left(\bigcap_{\gamma \in S} U_{1,\gamma}' \right).$$

Suppose $\gamma \in SL_2(\mathbb{Z})$ s.t. $\gamma U \cap U' \neq \emptyset$. WTS $\gamma\tau = \tau'$; suffices to check $\gamma \notin S$. Indeed if $\gamma \in S$, then

$$\emptyset = U_{1,\gamma} \cap U_{1,\gamma}' \supseteq \gamma U \cap U' \neq \emptyset$$

contradiction $\square$

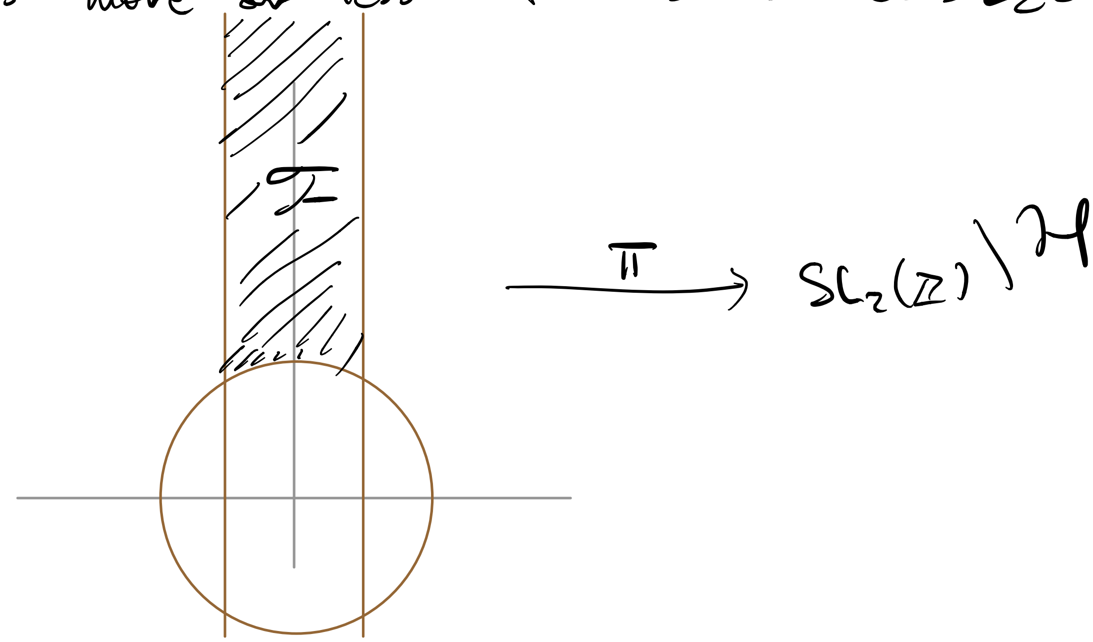
$\square$

Fundamental domains

For now, $\Gamma = SL_2(\mathbb{Z})$. We will see that

$$\mathcal{F} = \{\tau \in \mathcal{H} : |\mathrm{Re}(\tau)| \le 1/2, \quad |\tau| \ge 1\} \subseteq \mathcal{H}$$

is "more or less" the same as $SL_2(\mathbb{Z}) \backslash \mathcal{H}$.



Prop

(1) $\forall \tau \in \mathcal{H}$, $\exists \gamma \in SL_2(\mathbb{Z})$ s.t. $\gamma\tau \in \mathcal{F}$

(2) If $\tau \neq \tau' \in \mathcal{F}$ with $\gamma\tau = \tau'$ some $I \neq \gamma \in SL_2(\mathbb{Z})$ then either:

(i) $\mathrm{Re}(\tau) = \pm 1/2$, $\quad \tau' = \tau \mp 1$

(ii) $|\tau| = 1$, $\quad \tau' = \frac{-1}{\tau}$

(We call $\mathcal{F}$ a "fundamental domain" for $SL_2(\mathbb{Z}) \backslash \mathcal{H}$.)

<u>Pf.</u> (1) Let $\tau \in \mathcal{H}$. By applying $T^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ for some $k \in \mathbb{Z}$, can move τ into $|\mathrm{Re}(\tau)| \leq 1/2$. If $|\tau| \geq 1$, we're done, so suppose $|\tau| < 1$.

Note that with $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, we have $\mathrm{Im}(S\tau) = \dfrac{\mathrm{Im}(\tau)}{|\tau|^2} > \mathrm{Im}(\tau)$. Thus by applying S and then T^k, some k, get τ_1 with $\tau_1 \in \mathcal{F}$ or $|\tau_1| < 1$ with $\mathrm{Im}(\tau_1) > \tau$. We can iterate this to either get some $\gamma\tau \in \mathcal{F}$ or an infinite set $\{\tau_1, \tau_2, \tau_3, \dots\}$ in $\{|\tau| < 1\}$ with $\mathrm{Im}(\tau_1) < \mathrm{Im}(\tau_2) < \cdots$.

But the latter is impossible: the set of $(c,d) \in \mathbb{Z}^2$ with $|c\tau + d| < 1$ is finite and since

$$\mathrm{Im}(\gamma\tau) = \frac{\mathrm{Im}(\tau)}{|c\tau + d|^2}$$

so there is a $\max\{\mathrm{Im}(\gamma\tau) : \gamma \in SL_2(\mathbb{Z})\}$, contradiction.

(2) Exercise. $\square$

Note: This gives fundamental domains for $\Gamma\backslash\mathcal{H}$ for congruence subgps $\Gamma \leq SL_2(\mathbb{Z})$: if $\{\gamma_1, \dots, \gamma_n\}$ is a set of reps for

$\Gamma\backslash SL_2(\mathbb{Z})$ then

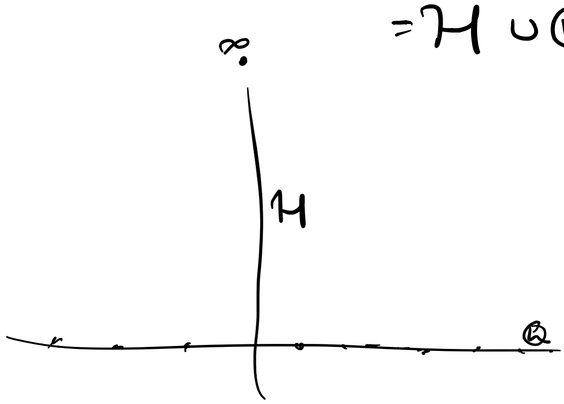
$$\bigcup_{\gamma_i} \gamma_i \mathcal{F} \longrightarrow \Gamma\backslash\mathcal{H}$$

is a surjection, and the only pts identified are bdry pts.

Compactification

We would like to put $\Gamma\backslash\mathcal{H}$ inside a compact space with a R.S. structure which turns $\Gamma\backslash\mathcal{H}$ into an open subspace.

Defn Let $\mathcal{H}^* = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$
$= \mathcal{H} \cup \mathbb{Q} \cup \{\infty\}$.



Note: $SL_2(\mathbb{Z})$ acts on $\mathbb{P}^1(\mathbb{Q})$ in the usual way

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\left(\frac{p}{q}\right) = \frac{a(p/q)+b}{c(p/q)+d} = \frac{ap+bq}{cp+dq}$$

So
$$\tau \longmapsto \frac{a\tau+b}{c\tau+d}$$
$$\infty \longmapsto \frac{a}{c}$$
$$\frac{-d}{c} \longmapsto \infty \quad \text{if } c \neq 0$$

Also note: if $\frac{a}{c} \in \mathbb{Q}$ with $(a,c)=1$ then $\exists \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ so $\frac{a}{c} = \gamma \cdot \infty$ for some $\gamma \in SL_2(\mathbb{Z})$. So $SL_2(\mathbb{Z})$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$.

What is the topology on $\mathcal{H}^*$?

Start by defining a basis of opens at ∞ to be
$$U_r = \{\tau \in \mathcal{H} : \operatorname{Im}(\tau) > r\} \cup \{\infty\}$$
for $r > 0$.

Since we want $\gamma \in SL_2(\mathbb{Z})$ to act as a homeomorphism, this forces a basis at $q \in \mathbb{Q}$ to be $\{\gamma U_r\}_{r>0}$ for $q = \gamma\infty$.

γU_r are circles tangent to q!

Prop $\Gamma\backslash\mathcal{H}^*$ is a connected, compact, Hausdorff space.

Pf. connected: ✓

compact: $\mathcal{F}^* = \mathcal{F} \cup \{\infty\}$

is a fund. domain for $SL_2(\mathbb{Z})\backslash\mathcal{H}^*$.

so $\displaystyle\bigcup_{\gamma_i} \gamma_i \mathcal{F}^*$ is a fund. domain

for $\Gamma\backslash\mathcal{H}^*$, $\{\gamma_i\}$ reps for $\Gamma\backslash SL_2(\mathbb{Z})$.

$\mathcal{F}^*$ is compact!

So $\Gamma\backslash\mathcal{H}^*$ is a finite union of compact sets, hence compact. ✓

Hausdorff: exercise. □